Destruction of recursive trees

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ABSTRACT: *We study, for the family of recursive trees, two procedures that destroy trees by successively removing edges. In both variants, one starts with a tree T of size n and chooses one of the $n-1$ edges at random. Removing this edge costs a toll depending on the size of T, given by the toll function t_n and leads to two subtrees T' and T''. In the one-sided variant, the edge-removal procedure will be iterated with the subtree containing the root, whereas in the two-sided variant it will be iterated with both subtrees. For both variants, we study for toll functions $t_n = n^\alpha$ with $\alpha \geq 0$ the total costs (= sum of the tolls of every step) obtained by completely destroying random recursive trees, where we compute for this quantity the asymptotic behaviour of all moments.*

1 Introduction

In this paper, we are considering two recursive edge-removal procedures **P1** ("one-sided destructions") and **P2** ("two-sided destructions") to destroy (rooted) trees. Both variants start with a tree T of size $|T| = n$, where the size measures as usual the number of nodes of T. If $n = 1$ there are no edges that can be removed and both procedures **P1** and **P2** stop, but we assume that this costs the toll t_1. If $n \geq 2$, then one of the $n-1$ edges in the tree will be chosen and afterwards this edge will be removed from T. We assume now that removing this edge costs a certain toll depending on the size of T and which is given by the toll function t_n. After removing this edge, the original tree T falls into two subtrees T' and T'' with sizes $1 \leq |T'|, |T''| \leq n-1$, where one of them (let us assume T') contains the root of T. In the two-sided variant **P2**, the edge-removal procedure will now be applied recursively to both subtrees T' and T'', whereas in the one-sided variant **P1**, the edge-removal procedure (in [7] called "cutting-down") will only be applied to the subtree T' that contains the root.

Thus the procedure **P2** terminates and T has been destroyed by **P2**, when all $n-1$ edges are removed from T, whereas **P1** terminates and thus T has been destroyed by **P1**, when the root of T has been isolated. We are now interested in the total costs (= sum of the costs of every edge-removal step) $C_1(T)$ resp. $C_2(T)$ that occur when destroying a tree T by **P1** resp. **P2**. Of course, these quantities are for $|T| \geq 2$ given recursively by

$$C_1(T) = C_1(T') + t_{|T|}, \quad \text{resp.} \quad C_2(T) = C_2(T') + C_2(T'') + t_{|T|}, \tag{1}$$

where T', T'' are the subtrees appearing after the first edge-removal step and T' contains the root of T. If $|T| = 1$ then $C_1(T) = C_2(T) = t_1$. An example for destroying a tree by **P1** resp. **P2** is given in Figure 1.1.

In this paper, we study for toll functions $t_n = n^\alpha$ with $\alpha \geq 0$ the random variables X_n (resp. Y_n), which measure the total costs that accumulate when destroying a *random* recursive tree of size n by the *random* edge-removal procedures **P1** (resp. **P2**). The tree family considered and the probability model used are described next.

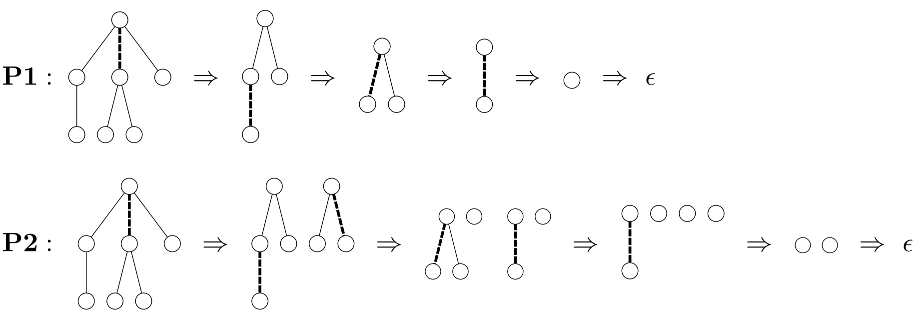
The family of recursive trees can be defined in the following way. A rooted labelled tree T of size n with labels $1, 2, \ldots, n$ is a recursive tree, if the root is labelled with 1, and for each node v holds that the labels of the vertices on the unique path from the root to v form an increasing sequence. It is seen easily that there are $T_n = (n-1)!$ different size-n recursive trees. As the model of randomness we use the random tree model, which means that every recursive tree of size n can be chosen as input for the edge-removal procedures with equal probability $\frac{1}{(n-1)!}$. We speak then about random recursive trees or uniform recursive trees. For a survey of applications and results on random recursive trees see [5].

Further, we will always assume for **P1** resp. **P2** that the removed edges are at each stage chosen at random from the remaining tree. We speak thus about the *random* edge-removal procedures.

If one chooses the toll function $t_n = 1$ for $n \geq 2$ with $t_1 = 0$, then X_n measures exactly the number of edges that are removed by the cutting-down procedure **P1** to destroy a random size-n tree. This quantity was studied by Meir and Moon in [7], where they obtained the following results for the first two moments: $\mathbb{E}(X_n) \sim \frac{n}{\log n}$ and $\mathbb{E}(X_n^2) \sim \frac{n^2}{\log^2 n}$. Choosing the toll function $t_n = 1$, the (corresponding) quantity X_n was also studied for other tree families, see e. g. [6, 8]. For unrooted labelled trees, the (corresponding) quantity Y_n was studied for a few toll functions t_n in [4] and for general toll functions $t_n = n^\alpha$ in [1].

Figure 1.1

Destruction of a tree T of size 7 by the procedures **P1** *and* **P2**. *Using the toll function $t_n = n$ for $n \geq 1$, the one-sided destruction has total costs $C_1(T) = 17$ and the two-sided destruction has total costs $C_2(T) = 28$. Here, ϵ denotes the empty tree.*



2 Results and mathematical preliminaries

We analyzed the behaviour of the moments $\mathbb{E}(X_n^s)$ resp. $\mathbb{E}(Y_n^s)$ of the total costs when destroying size-n recursive trees with procedures **P1** resp. **P2** for toll functions $t_n = n^\alpha$ with $\alpha \geq 0$ resp. $\alpha > 0$ ($\alpha = 0$ for Y_n is trivial, since then $Y_n = 2n-1$) and have obtained the following results. Here $H_n := \sum_{k=1}^{n} \frac{1}{k}$ denotes as usual the n-th harmonic number and $\Psi(x) := \frac{d}{dx} \log \Gamma(x)$ denotes the Psi-function.

Theorem 2.1 *The s-th moments $\mathbb{E}(X_n^s)$ resp. s-th centered moments $\mathbb{E}\big([X_n - \mathbb{E}(X_n)]^s\big)$ of the total costs X_n incurred by one-sided destructions of random recursive trees for toll functions $t_n = n^\alpha$ with $\alpha \geq 0$ are, for s an integer and*

$n \to \infty$, *asymptotically given by*

$$\mathbb{E}\big(X_n^s\big) = \frac{1}{(\alpha+1)^s}\frac{n^{s(\alpha+1)}}{\log^s n} + \frac{H_s + s + (\alpha+1)\sum_{l=1}^{s}\Psi\big(l(\alpha+1)\big)}{(\alpha+1)^{s+1}}\frac{n^{s(\alpha+1)}}{\log^{s+1} n} \tag{2}$$
$$+ \mathcal{O}\Big(\frac{n^{s(\alpha+1)}}{\log^{s+2} n}\Big), \ s \geq 1,$$

$$\mathbb{E}\big([X_n - \mathbb{E}(X_n)]^s\big) = \frac{\frac{(-1)^{s-1}}{(\alpha+1)s} + \sum\limits_{l=0}^{s-1}\binom{s-1}{l}(-1)^{s-1-l}\Psi\big((l+1)(\alpha+1)\big)}{(\alpha+1)^s}\frac{n^{s(\alpha+1)}}{\log^{s+1} n}$$
$$+ \mathcal{O}\Big(\frac{n^{s(\alpha+1)}}{\log^{s+2} n}\Big), \ s \geq 2. \tag{3}$$

Theorem 2.2 *The s-th moments $\mathbb{E}\big(Y_n^s\big)$ resp. s-th centered moments $\mathbb{E}\big([Y_n - \mathbb{E}(Y_n)]^s\big)$ of the total costs Y_n incurred by two-sided destructions of random recursive trees for toll functions $t_n = n^\alpha$ with $\alpha > 0$ are, for s an integer and $n \to \infty$, asymptotically given by*

$$\mathbb{E}\big(Y_n^s\big) = \frac{1}{(\alpha+1)^s}\frac{n^{s(\alpha+1)}}{\log^s n} + \gamma_s\frac{n^{s(\alpha+1)}}{\log^{s+1} n} + \mathcal{O}\Big(\frac{n^{s(\alpha+1)}}{\log^{s+2} n}\Big), \ s \geq 1, \tag{4}$$

$$\mathbb{E}\big([Y_n - \mathbb{E}(Y_n)]^s\big) = \delta_s\frac{n^{s(\alpha+1)}}{\log^{s+1} n} + \mathcal{O}\Big(\frac{n^{s(\alpha+1)}}{\log^{s+2} n}\Big), \ s \geq 2, \tag{5}$$

where the appearing constants γ_s and δ_s are given by

$$\gamma_s = \frac{\frac{s}{\alpha+1} + \sum\limits_{l=1}^{s}\Psi\big(l(\alpha+1)\big) + \sum\limits_{l=1}^{s}\frac{1}{l(\alpha+1)-1} + \sum\limits_{l=1}^{s}\sum\limits_{j=1}^{l-1}\binom{l}{j}\frac{\Gamma\big(j(\alpha+1)+1\big)\Gamma\big((l-j)(\alpha+1)-1\big)}{\Gamma\big(l(\alpha+1)+1\big)}}{(\alpha+1)^s},$$

$$\delta_s = \frac{1}{(\alpha+1)^s}\Bigg[\sum_{l=0}^{s-1}\binom{s-1}{l}(-1)^{s-1-l}\Psi\big((l+1)(\alpha+1)\big) + \frac{(-1)^{s-1}(s-1)!(\alpha+1)^{s-1}}{\prod_{l=1}^{s}(l(\alpha+1)-1)}$$
$$+ \sum_{l=1}^{s}\binom{s-1}{l-1}(-1)^{s-l}\sum_{j=1}^{l-1}\binom{l}{j}\frac{\Gamma\big(j(\alpha+1)+1\big)\Gamma\big((l-j)(\alpha+1)-1\big)}{\Gamma\big(l(\alpha+1)+1\big)}\Bigg].$$

From these theorems follow that for the toll $t_n = n^\alpha$ the r. v. $\frac{(\alpha+1)\log n}{n^{\alpha+1}}X_n$ resp. $\frac{(\alpha+1)\log n}{n^{\alpha+1}}Y_n$ converge in probability to 1 with convergence of all moments. Furthermore, if W_n (resp. $\tilde{W}_n$) denotes a zero-mean and unit-variance normalization of X_n (resp. Y_n), then W_n (resp. $\tilde{W}_n$) has s-th moments growing like $\log^{\frac{s}{2}-1} n$ for $s \geq 2$. This shows that if W_n (resp. $\tilde{W}_n$) has a limiting distribution, this cannot be established by the method of moments. It remains still open whether there exists a limiting distribution for some centered and scaled version of X_n (resp. Y_n). Also it seems surprising, that we do not have a lead-order discrepancy between X_n and Y_n, although it holds of course $Y_n \geq X_n$.

To prove Theorem 2.1 and Theorem 2.2, we use a recursive approach, as it was done for one-sided destructions and the special toll function $t_n = 1$ in [7]. That this recursive approach is indeed permitted is stated in Lemma 3.1. The appearing distribution recurrences (6) lead to recurrences (8) and (12) for the s-th moments. Using generating functions, we obtain differential equations for every s-th moment that can be solved, where the solutions (11) and (15) are composed of the operations differentiation, integration and the Hadamard product of generating functions of the lower moments and the toll function. The Hadamard product $F(z) \odot G(z)$ of generating functions $F(z) = \sum_{n \geq 0} f_n z^n$ and $G(z) = \sum_{n \geq 0} g_n z^n$ is defined by $F(z) \odot G(z) := \sum_{n \geq 0} f_n g_n z^n$. Moreover we use in this paper the abbreviation $F^{\odot s}(z) := \underbrace{F(z) \odot \cdots \odot F(z)}_{s \text{ times}} = \sum_{n \geq 0} f_n^s z^n$.

To extract the asymptotic information from the solutions (11) and (15) we cannot use the extension of the "singularity-analysis-toolbox" given in [1], since the theorems shown therein deal only with positive integral powers of logarithmic terms whereas here occur negative powers. Thus we will go back to a quite elementary, but here efficient approach that computes the asymptotic growth directly at the level of the coefficients. This has the advantage that the effect of the operations integration, differentiation and Hadamard product to the growth of the coefficients can be described easily, whereas of course difficulties may arise when computing the growth of the coefficients of the Cauchy product. But for the problem considered here, the two summation formulæ (16) and (17) are sufficient. Using them, the computations for one-sided destructions are done in Section 5 and (only sketched) for two-sided destructions in Section 6.

3 Recurrences for the quantities considered

The basic idea in our approach is to study the distribution recurrences

$$X_n \stackrel{\mathcal{L}}{=} X_{K_n} + t_n, \; n \geq 2, \; X_1 = t_1; \quad Y_n \stackrel{\mathcal{L}}{=} Y_{K_n} + \tilde{Y}_{n-K_n} + t_n, \; n \geq 2, \; Y_1 = t_1, \quad (6)$$

where Y_n and $\tilde{Y}_n$ are identically and independently distributed random variables and K_n is independent of X_n, Y_n and $\tilde{Y}_n$. The random variable K_n will be given by the splitting probabilities $p_{n,k} := \mathbb{P}\{K_n = k\}$, for $1 \leq k \leq n-1$, where $p_{n,k}$ is the probability that after removing a random edge from a size-n random recursive tree, the subtree containing the root has size k.

To reduce this problem to a study of (6), it is of course necessary that randomness is preserved by cutting-off a random edge. This means that after removing a randomly selected edge from a size-n random recursive tree, the remaining subtrees with sizes k resp. $n-k$ are after natural-order-preserving relabellings of the nodes with labels $\{1, \ldots, k\}$ resp. $\{1, \ldots, n-k\}$ again *random* recursive trees of sizes k resp. $n-k$. This property of random recursive trees was shown implicitly in [7] when computing the splitting probabilities $p_{n,k}$. Since this is a crucial point in our approach, we will restate their proof. Thus randomness is actually preserved by cutting-off a random edge and the recurrences (6) follow directly from (1).

Lemma 3.1 (Meir and Moon, 1974) *Let us assume that we choose a random recursive tree T of size n and also one of its $n-1$ edges at random, and after removing this edge, the remaining subtrees T' resp. T'' are of sizes k resp. $n-k$, where we further assume that T' contains the root of T. Then it holds that, after an order-preserving relabelling of the nodes, both subtrees are random recursive trees of sizes k resp. $n-k$ and the splitting probabilities $p_{n,k}$ are given by*

$$p_{n,k} = \frac{n}{(n-1)(n-k)(n-k+1)}, \quad \textit{for } 1 \leq k \leq n-1. \quad (7)$$

Proof. Starting with a size-n recursive tree and removing one of its $n-1$ edges, we obtain a subtree T' of size $1 \le k \le n-1$ which contains the root of T and another subtree T'' of size $n-k$. After the order-preserving relabellings, we can consider both subtrees as recursive trees. Now we want to count, how often we can obtain a particular pair (T', T'') of recursive trees with sizes k resp. $n-k$, when removing one edge of recursive trees of size n. It will turn out that this quantity $w(T', T'')$ depends only on the sizes k and n, not on the particular chosen trees T' and T'' and the lemma will be proven. Equivalently we can go the other way around and ask, in how many ways $w(T', T'')$ can we reconstruct size-n recursive trees from the pair (T', T''). Let us assume that the removed edge originally connected the root of T'' with the node with label j in T'. Then all $n-k$ nodes in T'' must have labels larger than j. We have thus $\binom{n-j}{n-k}$ possibilities to select them from $\{j+1, \dots, n\}$ and distribute them order preserving to T'', whereas the remaining $k-j$ labels from $\{j+1, \dots, n\}$ are distributed order preserving to the nodes of T' with labels larger than j. This gives in that case $\binom{n-j}{n-k}$ different size-n recursive trees. By summing up we find that independently from the pair T' and T'', we always have $w(T', T'') = \sum_{j=1}^{k} \binom{n-j}{n-k} = \binom{n}{k-1}$ and therefore randomness is preserved.

Due to the given bijection between pairs of recursive trees with sizes k and $n-k$ and the pairs consisting of a size-n recursive tree and one of its $n-1$ edges, we obtain the equations $\sum_{k=1}^{n-1} \binom{n}{k-1} T_k T_{n-k} = (n-1) T_n$ and $p_{n,k} = \binom{n}{k-1} \frac{T_k T_{n-k}}{(n-1)T_n}$. Thus (7) is also shown.

□

From the distribution recurrence (6), we obtain then the following recurrences for the s-th moments $\mu_n^{[s]} := \mathbb{E}\big(X_n^s\big)$ of X_n:

$$\mu_n^{[s]} = \mathbb{E}\big([X_{K_n} + t_n]^s\big) = \sum_{s_1+s_2=s} \binom{s}{s_1} t_n^{s_1} \mathbb{E}\big(X_{K_n}^{s_2}\big) = \sum_{s_1+s_2=s} \binom{s}{s_1} t_n^{s_1} \sum_{k=1}^{n-1} p_{n,k} \mu_k^{[s_2]}.$$

We write them as

$$\mu_n^{[s]} = \sum_{k=1}^{n-1} p_{n,k} \mu_k^{[s]} + r_n^{[s]}, \quad \text{for } n \ge 2, \qquad \mu_1^{[s]} = t_1^s, \tag{8}$$

with $r_n^{[s]} = \sum_{\substack{s_1+s_2=s,\\ s_2<s}} \binom{s}{s_1} t_n^{s_1} \sum_{k=1}^{n-1} p_{n,k} \mu_k^{[s_2]}$.

Introducing the generating functions resp. the common abbreviation

$$\mu^{[s]}(z) := \sum_{n\ge1} \mu_n^{[s]} \frac{z^n}{n}, \;\; t(z) := \sum_{n\ge1} t_n z^n, \;\; r^{[s]}(z) := \sum_{n\ge2} (n-1) r_n^{[s]} \frac{z^n}{n}, \;\; L(z) := \log \frac{1}{1-z},$$

we obtain from (8) by multiplying with $\frac{n-1}{n} z^n$ and summing up:

$$\sum_{n\ge2} \mu_n^{[s]} \frac{n-1}{n} z^n = z \frac{d}{dz} \mu^{[s]}(z) - \mu^{[s]}(z),$$

$$\sum_{n\ge2} \sum_{k=1}^{n-1} \frac{1}{(n-k)(n-k+1)} \mu_k^{[s]} z^n = \big(\frac{d}{dz} \mu^{[s]}(z)\big)\big(z - (1-z)L(z)\big),$$

$$r^{[s]}(z) = \sum_{\substack{s_1+s_2=s,\\ s_2<s}} \tbinom{s}{s_1} t^{\odot s_1}(z) \odot \Big[\big(\frac{d}{dz} \mu^{[s_2]}(z)\big)\big(z - (1-z)L(z)\big)\Big]. \tag{9}$$

This gives thus the differential equation

$$(1-z)L(z)\frac{d}{dz}\mu^{[s]}(z)-\mu^{[s]}(z)=r^{[s]}(z), \tag{10}$$

where $r^{[s]}(z)$ is given by (9). Solving and adapting to the initial condition $\frac{d}{dz}\mu^{[s]}(z)\big|_{z=0}=\mu_1^{[s]}=t_1^s$, we obtain the solution

$$\mu^{[s]}(z)=L(z)\int_{t=0}^{z}\frac{r^{[s]}(t)}{(1-t)L^2(t)}dt+t_1^sL(z). \tag{11}$$

Analogously we obtain from the distribution recurrence (6) the following recurrences for the s-th moments $\lambda_n^{[s]}:=\mathbb{E}\big(Y_n^s\big)$ of Y_n:

$$\lambda_n^{[s]}=\mathbb{E}\big([Y_{K_n}+\tilde{Y}_{n-K_n}+t_n]^s\big)=\sum_{s_1+s_2+s_3=s}\binom{s}{s_1,s_2,s_3}t_n^{s_1}\sum_{k=1}^{n-1}p_{n,k}\lambda_k^{[s_2]}\lambda_{n-k}^{[s_3]}.$$

We write them as

$$\lambda_n^{[s]}=\sum_{k=1}^{n-1}p_{n,k}(\lambda_k^{[s]}+\lambda_{n-k}^{[s]})+r_n^{[s]},\quad \text{for } n\ge 2,\qquad \lambda_1^{[s]}=t_1^s, \tag{12}$$

with $r_n^{[s]}=\sum_{\substack{s_1+s_2+s_3=s,\\ s_2,s_3<s}}\binom{s}{s_1,s_2,s_3}t_n^{s_1}\sum_{k=1}^{n-1}p_{n,k}\lambda_k^{[s_2]}\lambda_{n-k}^{[s_3]}$.

With the generating function $\lambda^{[s]}(z):=\sum_{n\ge1}\lambda_n^{[s]}\frac{z^n}{n}$, we obtain from (12) by multiplying with $\frac{n-1}{n}z^n$ and summing up:

$$\sum_{n\ge2}\lambda_n^{[s]}\frac{n-1}{n}z^n=z\frac{d}{dz}\lambda^{[s]}(z)-\lambda^{[s]}(z),$$

$$\sum_{n\ge2}\sum_{k=1}^{n-1}\frac{1}{(n-k)(n-k+1)}\lambda_k^{[s]}z^n=\big(\frac{d}{dz}\lambda^{[s]}(z)\big)\big(z-(1-z)L(z)\big),$$

$$\sum_{n\ge2}\sum_{k=1}^{n-1}\frac{1}{(n-k)(n-k+1)}\lambda_{n-k}^{[s]}z^n=\frac{1}{1-z}\int_{t=0}^{z}\lambda^{[s]}(t)dt,$$

$$r^{[s]}(z)=\sum_{\substack{s_1+s_2+s_3=s,\\ s_2,s_3<s}}\tbinom{s}{s_1,s_2,s_3}t^{\odot s_1}(z)\odot\Big[\big(\frac{d}{dz}\lambda^{[s_2]}(z)\big)\int_{t=0}^{z}\lambda^{[s_3]}(t)dt\Big]. \tag{13}$$

This leads to the following equation with $r^{[s]}(z)$ given by (13):

$$(1-z)L(z)\frac{d}{dz}\lambda^{[s]}(z)-\lambda^{[s]}(z)-\frac{1}{1-z}\int_{t=0}^{z}\lambda^{[s]}(t)dt=r^{[s]}(z). \tag{14}$$

By introducing the functions $\tilde{\lambda}^{[s]}(z):=\frac{d}{dz}\lambda^{[s]}(z)=\sum_{n\ge1}\lambda_n^{[s]}z^{n-1}$, this equation can be transformed to the first order differential equation

$$\frac{d}{dz}\tilde{\lambda}^{[s]}(z)-\frac{2}{1-z}\tilde{\lambda}^{[s]}(z)=\frac{1}{(1-z)^2L(z)}\frac{d}{dz}\big((1-z)r^{[s]}(z)\big),$$

which has by adapting to the initial condition $\tilde{\lambda}^{[s]}(0) = \lambda_1^{[s]} = t_1^s$ the solution

$$\tilde{\lambda}^{[s]}(z) = \frac{1}{(1-z)^2} \int_{t=0}^{z} \frac{1}{L(t)} \frac{d}{dt}\big((1-t) r^{[s]}(t)\big) dt + \frac{t_1^s}{(1-z)^2}. \tag{15}$$

4 Summation lemmata

Essential to our approach are the following auxiliary formulæ to control the asymptotic growth of the appearing convolutions.

Lemma 4.1 *For $\alpha, \beta, p, q \in \mathbb{R}$ with $\alpha, \beta > -1$ and $p, q \geq 0$ the following expansions hold:*

$$\begin{aligned}\sum_{k=2}^{n-2} \frac{k^\beta\,(n-k)^\alpha}{\log^q(k)\,\log^p(n-k)} &= \frac{n^{\alpha+\beta+1}}{\log^{p+q} n} \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} \times \\ &\times \left(1 - \frac{q\,\Psi(\beta+1) + p\,\Psi(\alpha+1) - (p+q)\Psi(\alpha+\beta+2)}{\log n} + \mathcal{O}\Big(\frac{1}{\log^2 n}\Big)\right),\end{aligned} \tag{16}$$

$$\sum_{k=2}^{n-2} \frac{(n-k)^\alpha}{k(k-1)\log^p(n-k)} = \frac{n^\alpha}{\log^p n} + \mathcal{O}\Big(\frac{n^{\alpha-1}}{\log^{p-2} n}\Big) + \mathcal{O}\Big(\frac{\log^2 n}{n}\Big). \tag{17}$$

Proof. To obtain (16), we start with the following expansion:

$$\begin{aligned}\sum_{k=2}^{n-2} \frac{k^\beta (n-k)^\alpha}{\log^q(k)\log^p(n-k)} &= \frac{n^{\alpha+\beta}}{\log^{p+q} n} \sum_{k=\lfloor \log n \rfloor}^{n-\lfloor \log n \rfloor} \frac{\big(\frac{k}{n}\big)^\beta \big(1-\frac{k}{n}\big)^\alpha}{\big(1+\frac{\log \frac{k}{n}}{\log n}\big)^q \big(1+\frac{\log(1-\frac{k}{n})}{\log n}\big)^p} \\ &+ \sum_{k=2}^{\lfloor \log n \rfloor - 1} \frac{k^\beta (n-k)^\alpha}{\log^q(k)\log^p(n-k)} + \sum_{k=n+1-\lfloor \log n \rfloor}^{n-2} \frac{k^\beta (n-k)^\alpha}{\log^q(k)\log^p(n-k)} \\ &= \frac{n^{\alpha+\beta+1}}{\log^{p+q} n} \sum_{k=\lfloor \log n \rfloor}^{n-\lfloor \log n \rfloor} \frac{1}{n}\Big(\frac{k}{n}\Big)^\beta \Big(1-\frac{k}{n}\Big)^\alpha \times \Big[1 - \frac{q\log\frac{k}{n} + p\log(1-\frac{k}{n})}{\log n} + \mathcal{O}\Big(\frac{\log^2\frac{k}{n}}{\log^2 n}\Big) \\ &+ \mathcal{O}\Big(\frac{\big(\log\frac{k}{n}\big)\big(\log(1-\frac{k}{n})\big)}{\log^2 n}\Big) + \mathcal{O}\Big(\frac{\log^2(1-\frac{k}{n})}{\log^2 n}\Big)\Big] + \mathcal{O}\big(n^\alpha (\log n)^{\beta+1-p}\big) + \mathcal{O}\big(n^\beta (\log n)^{\alpha+1-q}\big).\end{aligned}$$

The appearing sums can be considered as Riemann sums for the Beta integral resp. their derivatives. As examples, we give the following two computations:

$$\begin{aligned}&\sum_{k=\lfloor \log n \rfloor}^{n-\lfloor \log n \rfloor} \frac{1}{n}\Big(\frac{k}{n}\Big)^\beta \Big(1-\frac{k}{n}\Big)^\alpha = \int_{x=0}^{1} x^\beta (1-x)^\alpha dx + \mathcal{O}\Big(\frac{(\log n)^\beta}{n^{\beta+1}}\Big) + \mathcal{O}\Big(\frac{(\log n)^\alpha}{n^{\alpha+1}}\Big) \\ &= \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} + \mathcal{O}\Big(\frac{(\log n)^\beta}{n^{\beta+1}}\Big) + \mathcal{O}\Big(\frac{(\log n)^\alpha}{n^{\alpha+1}}\Big), \\ &\sum_{k=\lfloor \log n \rfloor}^{n-\lfloor \log n \rfloor} \frac{1}{n}\Big(\frac{k}{n}\Big)^\beta \Big(1-\frac{k}{n}\Big)^\alpha \log\frac{k}{n} = \int_{x=0}^{1} x^\beta (1-x)^\alpha \log x\, dx + \mathcal{O}\Big(\frac{(\log n)^{\beta+1}}{n^{\beta+1}}\Big) \\ &\quad + \mathcal{O}\Big(\frac{(\log n)^\alpha}{n^{\alpha+1}}\Big) = \frac{\partial}{\partial \beta}\Big(\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}\Big) + \mathcal{O}\Big(\frac{(\log n)^{\beta+1}}{n^{\beta+1}}\Big) + \mathcal{O}\Big(\frac{(\log n)^\alpha}{n^{\alpha+1}}\Big),\end{aligned}$$

where we obtain $\frac{\partial}{\partial\beta}\Big(\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}\Big) = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}\big(\Psi(\beta+1)-\Psi(\alpha+\beta+2)\big)$. Analogously one can treat the remaining sums which leads eventually to (16).

It remains to prove (17). We start with dissecting the summation interval

$$\sum_{k=2}^{n-2} \frac{(n-k)^\alpha}{k(k-1)\log^p(n-k)} = \sum_{k=2}^{\lfloor \frac{n}{\log n}\rfloor} \frac{(n-k)^\alpha}{k(k-1)\log^p(n-k)} + \sum_{k=\lfloor \frac{n}{\log n}\rfloor+1}^{n-2} \frac{(n-k)^\alpha}{k(k-1)\log^p(n-k)}$$

$$= \frac{n^\alpha}{\log^p n} \sum_{k=2}^{\lfloor \frac{n}{\log n}\rfloor} \frac{(1-\frac{k}{n})^\alpha}{k(k-1)\big(1+\frac{\log(1-\frac{k}{n})}{\log n}\big)^p} + \mathcal{O}\Big(\frac{n^{\alpha-1}}{\log^{p-2} n}\Big) + \mathcal{O}\Big(\frac{\log^2 n}{n}\Big),$$

where the remainder bounds are coming from the estimate $\frac{(n-k)^\alpha}{\log^p(n-k)} = \mathcal{O}(\frac{n^\alpha}{\log^p n}) + \mathcal{O}(1)$, which combines bounds for $\alpha > 0$ and $\alpha \le 0$ in the considered range $\lfloor \frac{n}{\log n}\rfloor < k \le n-2$. The remaining sum can be evaluated asymptotically which gives the main term in (17):

$$\sum_{k=2}^{\lfloor \frac{n}{\log n}\rfloor} \frac{(1-\frac{k}{n})^\alpha}{k(k-1)\big(1+\frac{\log(1-\frac{k}{n})}{\log n}\big)^p} = \sum_{k=2}^{\lfloor \frac{n}{\log n}\rfloor} \frac{1}{k(k-1)}\Big(1+\mathcal{O}(\frac{k}{n})\Big)$$
$$= \sum_{k=2}^{\lfloor \frac{n}{\log n}\rfloor} \frac{1}{k(k-1)} + \mathcal{O}\Big(\frac{\sum_{k=2}^{\lfloor \frac{n}{\log n}\rfloor}\frac{1}{k-1}}{n}\Big) = 1 + \mathcal{O}\Big(\frac{\log n}{n}\Big).$$

□

5 One-sided destructions

Now we want to prove the results of Theorem 2.1 concerning the asymptotic behaviour of the s-th moments of the total costs X_n when destroying random recursive trees with toll functions $t_n = n^\alpha$, for $n \ge 2$ with $\alpha \ge 0$ and $t_1 = 0$. Choosing $t_1 = 0$ instead of $t_1 = 1$ has of course no influence to the stated asymptotic behaviour ($X_n^* = X_n + 1$, if X_n^* measures the total costs with toll function $t_n = n^\alpha$, for $n \ge 1$), but (11) is then slightly simpler.

In order to reduce extracting coefficients from (11) to an application of formulæ (16) and (17) and to avoid dealing with convolutions of functions growing as n^{-1}, we introduce the generating functions $\tilde{\mu}^{[s]}(z) := \frac{d}{dz}\mu^{[s]}(z) = \sum_{n\ge1} \mu_n^{[s]} z^{n-1}$ and differentiate (11):

$$\tilde{\mu}^{[s]}(z) = \frac{r^{[s]}(z)}{(1-z)L(z)} + \frac{1}{1-z}\int_{t=0}^{z} \frac{r^{[s]}(t)}{(1-t)L^2(t)}dt, \qquad (18)$$

where $r^{[s]}(z)$ is given by (9) with there appearing functions $t(z) = \sum_{n\ge2} n^\alpha z^n$.

Now we want to show the expansion (2) by induction, where we additionally use, for $\beta, q > 0$, the asymptotic growth of the coefficients (see [2])

$$[z^n]\frac{1}{(1-z)^\beta\big(L(z)\big)^q} = \frac{n^{\beta-1}}{\Gamma(\beta)\log^q n}\Big(1+\frac{q\Psi(\beta)}{\log n}+\mathcal{O}(\frac{1}{\log^2 n})\Big). \qquad (19)$$

We further use the trivial effect to the growth of the coefficients when differentiating and integrating generating functions $F(z) = \sum_{n\geq 2} f_n z^n$ with $f_n = \frac{n^\beta}{\log^q n}$:

$$[z^n]\int_{t=0}^{z} F(t)dt = \frac{n^{\beta-1}}{\log^q n}\Big(1+\mathcal{O}\big(\tfrac{1}{n}\big)\Big), \quad [z^n]\frac{d}{dz}F(z) = \frac{n^{\beta+1}}{\log^q n}\Big(1+\mathcal{O}\big(\tfrac{1}{n}\big)\Big). \tag{20}$$

Since $\mu^{[0]}(z) = \sum_{n\geq 1}\frac{z^n}{n} = L(z)$ and $r^{[1]}(z) = t(z)\odot\big[\frac{1}{1-z} - 1 - L(z)\big]$, we obtain

$$[z^n]t(z) = n^\alpha, \;\; [z^n]\Big(\frac{1}{1-z} - 1 - L(z)\Big) = 1 - \frac{1}{n}, \;\; [z^n]r^{[1]}(z) = n^\alpha - n^{\alpha-1}, \;\; \text{for } n\geq 2.$$

Now we use (16) and (19) to obtain

$$\begin{aligned}
[z^n]\frac{r^{[1]}(z)}{(1-z)L(z)} &= \sum_{k=2}^{n-2}\Big(\tfrac{1}{\log k} + \tfrac{\Psi(1)}{\log^2 k} + \mathcal{O}\big(\tfrac{1}{\log^3 k}\big)\Big)\Big((n-k)^\alpha + \mathcal{O}\big((n-k)^{\alpha-1}\big)\Big) + \mathcal{O}\big(n^\alpha\big)\\
&= \sum_{k=2}^{n-2}\frac{(n-k)^\alpha}{\log k} + \Psi(1)\sum_{k=2}^{n-2}\frac{(n-k)^\alpha}{\log^2 k} + \mathcal{O}\Big(\sum_{k=2}^{n-2}\tfrac{(n-k)^\alpha}{\log^3 k}\Big) + \mathcal{O}\Big(\sum_{k=2}^{n-2}\tfrac{(n-k)^{\alpha-1}}{\log k}\Big) + \mathcal{O}\big(n^\alpha\big)\\
&= \frac{1}{\alpha+1}\frac{n^{\alpha+1}}{\log n} + \frac{\Psi(\alpha+2)}{\alpha+1}\frac{n^{\alpha+1}}{\log^2 n} + \mathcal{O}\Big(\frac{n^{\alpha+1}}{\log^3 n}\Big)
\end{aligned} \tag{21}$$

and

$$\begin{aligned}
[z^n]\frac{r^{[1]}(z)}{(1-z)L^2(z)} &= \sum_{k=2}^{n-2}\Big(\frac{1}{\log^2 k} + \mathcal{O}\big(\frac{1}{\log^3 k}\big)\Big)\Big((n-k)^\alpha + \mathcal{O}\big((n-k)^{\alpha-1}\big)\Big) + \mathcal{O}\big(n^\alpha\big)\\
&= \sum_{k=2}^{n-2}\frac{(n-k)^\alpha}{\log^2 k} + \mathcal{O}\Big(\sum_{k=2}^{n-2}\frac{(n-k)^\alpha}{\log^3 k}\Big) + \mathcal{O}\Big(\sum_{k=2}^{n-2}\frac{(n-k)^{\alpha-1}}{\log^2 k}\Big) + \mathcal{O}\big(n^\alpha\big)\\
&= \frac{1}{\alpha+1}\frac{n^{\alpha+1}}{\log^2 n} + \mathcal{O}\Big(\frac{n^{\alpha+1}}{\log^3 n}\Big).
\end{aligned}$$

We proceed with $[z^n]\int_{t=0}^{z}\frac{r^{[1]}(t)}{(1-t)L^2(t)}dt = \frac{1}{\alpha+1}\frac{n^\alpha}{\log^2 n} + \mathcal{O}\Big(\frac{n^\alpha}{\log^3 n}\Big)$ and another application of the convolution formula (16):

$$\begin{aligned}
[z^n]\frac{1}{1-z}\int_{t=0}^{z}\frac{r^{[1]}(t)}{(1-t)L^2(t)}dt &= \frac{1}{\alpha+1}\sum_{k=2}^{n-2}\frac{(n-k)^\alpha}{\log^2(n-k)} + \mathcal{O}\Big(\sum_{k=2}^{n-2}\tfrac{(n-k)^\alpha}{\log^3(n-k)}\Big)\\
&\quad + \mathcal{O}\Big(\tfrac{n^\alpha}{\log^2 n}\Big) = \frac{1}{(\alpha+1)^2}\frac{n^{\alpha+1}}{\log^2 n} + \mathcal{O}\Big(\frac{n^{\alpha+1}}{\log^3 n}\Big).
\end{aligned} \tag{22}$$

Adding (21) and (22), we obtain finally

$$\mu_{n+1}^{[1]} = [z^n]\tilde{\mu}^{[1]}(z) = \frac{1}{\alpha+1}\frac{n^{\alpha+1}}{\log n} + \frac{2+(\alpha+1)\Psi(\alpha+1)}{(\alpha+1)^2}\frac{n^{\alpha+1}}{\log^2 n} + \mathcal{O}\Big(\frac{n^{\alpha+1}}{\log^3 n}\Big).$$

Substituting $n-1$ for n, it follows that expansion (2) is valid for $s=1$.

Now we assume that for a given $s\geq 2$, all moments $\mathbb{E}(X_n^r) = \mu_n^{[r]}$ with $1\leq r < s$ have the expansion (2). We want to compute then the asymptotic growth of the coefficients $[z^n]r^{[s]}(z)$. To do this, we use $[z^n]t^{\odot s_1}(z) = n^{s_1\alpha}$ and the expansion given below which follows from (17) and (20):

$$[z^n]\big(\frac{d}{dz}\mu^{[s_2]}(z)\big)\big(z-(1-z)L(z)\big) =$$

- $1+\mathcal{O}\big(\frac{1}{n}\big)$, for $s_2 = 0$,
- $$\sum_{k=2}^{n-2}\frac{1}{k(k-1)}\Big[\frac{1}{(\alpha+1)^{s_2}}\frac{(n+1-k)^{s_2(\alpha+1)}}{\log^{s_2}(n+1-k)} + \frac{H_{s_2}+s_2+(\alpha+1)\sum_{l=1}^{s_2}\Psi\big(l(\alpha+1)\big)}{(\alpha+1)^{s_2+1}}\times$$
$$\times\frac{(n+1-k)^{s_2(\alpha+1)}}{\log^{s_2+1}(n+1-k)} + \mathcal{O}\Big(\frac{(n+1-k)^{s_2(\alpha+1)}}{\log^{s_2+2}(n+1-k)}\Big)\Big] + \mathcal{O}\Big(\frac{1}{n^2}\Big)$$
$$= \frac{1}{(\alpha+1)^{s_2}}\frac{n^{s_2(\alpha+1)}}{\log^{s_2} n} + \frac{H_{s_2}+s_2+(\alpha+1)\sum_{l=1}^{s_2}\Psi\big(l(\alpha+1)\big)}{(\alpha+1)^{s_2+1}}\frac{n^{s_2(\alpha+1)}}{\log^{s_2+1} n}$$
$$+\mathcal{O}\Big(\frac{n^{s_2(\alpha+1)}}{\log^{s_2+2} n}\Big), \text{ for } 1\le s_2 < s.$$

Thus for $s_1+s_2 = s$ and the restriction $0 \le s_2 < s$, we can describe the growth of the summands of $[z^n]r^{[s]}(z)$:

$$[z^n]\binom{s}{s_1}t^{\odot s_1}\odot\Big[\big(\frac{d}{dz}\mu^{[s_2]}(z)\big)\big(z-(1-z)L(z)\big)\Big] =$$

- $n^{s\alpha}+\mathcal{O}(n^{s\alpha-1})$, for $s_2 = 0$,
- $$\binom{s}{s_1}\frac{1}{(\alpha+1)^{s_2}}\frac{n^{s\alpha+s_2}}{\log^{s_2} n} + \binom{s}{s_1}\frac{H_{s_2}+s_2+(\alpha+1)\sum_{l=1}^{s_2}\Psi\big(l(\alpha+1)\big)}{(\alpha+1)^{s_2+1}}\frac{n^{s\alpha+s_2}}{\log^{s_2+1} n}$$
$$+\mathcal{O}\Big(\frac{n^{s\alpha+s_2}}{\log^{s_2+2} n}\Big), \text{ for } 1\le s_2 < s.$$

Therefore the maximum is obtained when choosing $s_2 = s-1$ and the other contributions are asymptotically negligible. This leads to the required expansion

$$[z^n]r^{[s]}(z) = \frac{s}{(\alpha+1)^{s-1}}\frac{n^{s(\alpha+1)-1}}{\log^{s-1} n}$$
$$+\frac{s\big[H_{s-1}+s-1+(\alpha+1)\sum_{l=1}^{s-1}\Psi\big(l(\alpha+1)\big)\big]}{(\alpha+1)^s}\frac{n^{s(\alpha+1)-1}}{\log^s n} + \mathcal{O}\Big(\frac{n^{s(\alpha+1)-1}}{\log^{s+1} n}\Big).$$

Then by applying (16) and (19), we obtain after some easy manipulations the following expansion:

$$[z^n]\frac{r^{[s]}(z)}{(1-z)L(z)} = \sum_{k=2}^{n-2}\Big(\frac{1}{\log k}+\frac{\Psi(1)}{\log^2 k}+\mathcal{O}\big(\frac{1}{\log^3 k}\big)\Big)\times$$
$$\times\Big[\frac{s}{(\alpha+1)^{s-1}}\frac{(n-k)^{s(\alpha+1)-1}}{\log^{s-1}(n-k)} + \frac{s\big[H_{s-1}+s-1+(\alpha+1)\sum_{l=1}^{s-1}\Psi\big(l(\alpha+1)\big)\big]}{(\alpha+1)^s}\times$$
$$\times\frac{(n-k)^{s(\alpha+1)-1}}{\log^s(n-k)} + \mathcal{O}\Big(\frac{(n-k)^{s(\alpha+1)-1}}{\log^{s+1}(n-k)}\Big)\Big] + \mathcal{O}\Big(\frac{n^{s(\alpha+1)-1}}{\log^{s-1} n}\Big)$$

$$= \frac{1}{(\alpha+1)^s}\frac{n^{s(\alpha+1)}}{\log^s n} + \frac{H_s + s - \frac{1}{s} + (\alpha+1)\sum_{l=1}^{s}\Psi\big(l(\alpha+1)\big)}{(\alpha+1)^{s+1}}\frac{n^{s(\alpha+1)}}{\log^{s+1} n} + \mathcal{O}\Big(\frac{n^{s(\alpha+1)}}{\log^{s+2} n}\Big). \tag{23}$$

Analogously we obtain

$$[z^n]\frac{r^{[s]}(z)}{(1-z)L^2(z)} = \frac{1}{(\alpha+1)^s}\frac{n^{s(\alpha+1)}}{\log^{s+1} n} + \mathcal{O}\Big(\frac{n^{s(\alpha+1)}}{\log^{s+2} n}\Big).$$

We proceed with $[z^n]\int_{t=0}^{z}\frac{r^{[s]}(t)}{(1-t)L^2(t)}dt = \frac{1}{(\alpha+1)^s}\frac{n^{s(\alpha+1)-1}}{\log^{s+1} n} + \mathcal{O}\Big(\frac{n^{s(\alpha+1)-1}}{\log^{s+2} n}\Big)$ and obtain again by using the convolution formula the expansion

$$[z^n]\frac{1}{1-z}\int_{t=0}^{z}\frac{r^{[s]}(t)}{(1-t)L^2(t)}dt = \sum_{k=2}^{n-2}\Big[\frac{1}{(\alpha+1)^s}\frac{(n-k)^{s(\alpha+1)-1}}{\log^{s+1}(n-k)} + \mathcal{O}\Big(\frac{(n-k)^{s(\alpha+1)-1}}{\log^{s+2}(n-k)}\Big)\Big]$$
$$+ \mathcal{O}\Big(\frac{n^{s(\alpha+1)-1}}{\log^{s+1} n}\Big) = \frac{1}{s(\alpha+1)^{s+1}}\frac{n^{s(\alpha+1)}}{\log^{s+1} n} + \mathcal{O}\Big(\frac{n^{s(\alpha+1)}}{\log^{s+2} n}\Big). \tag{24}$$

Adding (23) and (24), we obtain

$$\mu_{n+1}^{[s]} = [z^n]\tilde{\mu}^{[s]}(z) = \frac{1}{(\alpha+1)^s}\frac{n^{s(\alpha+1)}}{\log^s n}$$
$$+ \frac{H_s + s + (\alpha+1)\sum_{l=1}^{s}\Psi\big(l(\alpha+1)\big)}{(\alpha+1)^{s+1}}\frac{n^{s(\alpha+1)}}{\log^{s+1} n} + \mathcal{O}\Big(\frac{n^{s(\alpha+1)}}{\log^{s+2} n}\Big).$$

Substituting $n-1$ for n, it follows that expansion (2) is valid also for s.

To show the formula (3) for the centered moments, we plug in the asymptotic expansion (2) for the ordinary moments and get

$$\mathbb{E}\big([X_n - \mathbb{E}(X_n)]^s\big) = \sum_{k=0}^{s}\binom{s}{k}(-1)^{s-k}\,\mathbb{E}(X_n^k)\,\big(\mathbb{E}(X_n)\big)^{s-k} = \Bigg[\sum_{k=0}^{s}\frac{\binom{s}{k}(-1)^{s-k}}{(\alpha+1)^s}\Bigg]\frac{n^{s(\alpha+1)}}{\log^s n}$$
$$+ \Bigg\{\sum_{k=0}^{s}\frac{\binom{s}{k}(-1)^{s-k}}{(\alpha+1)^{s+1}}\Big[H_k + s\big(2 + (\alpha+1)\Psi(\alpha+1)\big) - k\big(1 + (\alpha+1)\Psi(\alpha+1)\big)$$
$$+ (\alpha+1)\sum_{l=1}^{k}\Psi\big(l(\alpha+1)\big)\Big]\Bigg\}\frac{n^{s(\alpha+1)}}{\log^{s+1} n} + \mathcal{O}\Big(\frac{n^{s(\alpha+1)}}{\log^{s+2} n}\Big).$$

Since it holds for $s \ge 1$ (see e. g. [3, p. 187ff]):

$$\sum_{k=0}^{s}\binom{s}{k}(-1)^{s-k}H_k = \frac{(-1)^{s-1}}{s},$$
$$\sum_{k=0}^{s}\binom{s}{k}(-1)^{s-k}\sum_{l=1}^{k}\Psi\big(l(\alpha+1)\big) = \sum_{l=0}^{s-1}\binom{s-1}{l}(-1)^{s-1-l}\Psi\big((l+1)(\alpha+1)\big),$$

and for $s \ge 2$: $\sum_{k=0}^{s}\binom{s}{k}(-1)^{s-k} = \sum_{k=0}^{s}\binom{s}{k}(-1)^{s-k}k = 0$, we obtain (3).

For the special case $\alpha = 0$, where X_n counts the number of removed edges until the root is isolated, we obtain due to further simplifications

$$\mathbb{E}(X_n^s) = \frac{n^s}{\log^s n} + \big((s+1)H_s - \gamma s\big)\frac{n^s}{\log^{s+1} n} + \mathcal{O}\Big(\frac{n^s}{\log^{s+2} n}\Big), \ \text{ for } s \ge 1,$$

$$\mathbb{E}\big([X_n - \mathbb{E}(X_n)]^s\big) = \frac{(-1)^s}{(s-1)s}\frac{n^s}{\log^{s+1} n} + \mathcal{O}\Big(\frac{n^s}{\log^{s+2} n}\Big), \text{ for } s \ge 2.$$

Here γ denotes as usual the Euler-constant.

We denote also the case $\alpha = 1$, where removing an edge of a tree T costs exactly the size $|T|$ of T:

$$\mathbb{E}(X_n^s) = \frac{1}{2^s}\frac{n^{2s}}{\log^s n} + \frac{\frac{1}{2}H_s + (2s+1)H_{2s} - (1+2\gamma)s}{2^{s+1}}\frac{n^{2s}}{\log^{s+1} n} + \mathcal{O}\Big(\frac{n^{2s}}{\log^{s+2} n}\Big), \text{ for } s \ge 1,$$

$$\mathbb{E}\big([X_n - \mathbb{E}(X_n)]^s\big) = \frac{(-1)^s}{2^{s+1}(s-1)s}\Big(1 + \frac{4^{s-1}}{\binom{2s-1}{s}}\Big)\frac{n^{2s}}{\log^{s+1} n} + \mathcal{O}\Big(\frac{n^{2s}}{\log^{s+2} n}\Big), \text{ for } s \ge 2.$$

We want to remark further that with the approach used here, one can also deal with logarithmic toll functions $t_n = \log^p n$ and compute the asymptotic behaviour of the moments $\mathbb{E}(X_n^s)$, where again one cannot deduce a limiting distribution.

6 Two-sided destructions

Here we sketch the proof of Theorem 2.2 concerning the asymptotic behaviour of the s-th moments of the total costs Y_n when destroying random recursive trees with toll functions $t_n = n^\alpha$, for $n \ge 2$ with $\alpha > 0$ and $t_1 = 0$. Again choosing $t_1 = 0$ instead of $t_1 = 1$ has no influence to the stated asymptotic behaviour ($Y_n^* = Y_n + n$, if Y_n^* measures the total costs with toll function $t_n = n^\alpha$, for $n \ge 1$).

To reduce extracting coefficients from (15) to an application of the convolution formula (16), we write equation (15) as

$$\begin{aligned}\tilde{\lambda}^{[s]}(z) = \frac{r^{[s]}(z)}{(1-z)L(z)} + \frac{1}{(1-z)^2L^2(z)}\int_{t=0}^{z} r^{[s]}(t)dt \\ + \frac{2}{(1-z)^2}\int_{t=0}^{z}\frac{1}{(1-t)L^3(t)}\int_{u=0}^{t} r^{[s]}(u)du\,dt,\end{aligned} \tag{25}$$

where $r^{[s]}(z)$ is given by (13) with there appearing functions $t(z) = \sum_{n\ge 2} n^\alpha z^n$.

To start the proof of the expansion (4) by induction, we consider $\tilde{\lambda}^{[0]}(z) = \sum_{n\ge 1}\frac{z^n}{n} = L(z)$ and obtain as in Section 5: $r^{[1]}(z) = t(z) \odot \big[\frac{1}{1-z} - 1 - L(z)\big]$ and we can use the expansion for $[z^n]r^{[1]}(z)$ given there. Also (21) is of interest.

Thus $[z^n]\int_{t=0}^{z} r^{[1]}(t)dt = n^{\alpha-1} + \mathcal{O}(n^{\alpha-2})$ and we obtain with (16) and (19)

$$[z^n]\frac{1}{(1-z)^2L^2(z)}\int_{t=0}^{z} r^{[1]}(t)dt = \frac{n^{\alpha+1}}{\alpha(\alpha+1)\log^2 n} + \mathcal{O}\Big(\frac{n^{\alpha+1}}{\log^3 n}\Big) \tag{26}$$

and by other two applications of the convolution formula also

$$[z^n]\frac{1}{(1-z)^2}\int_{t=0}^{z}\frac{1}{(1-t)L^3(t)}\int_{u=0}^{t} r^{[1]}(u)du\,dt = \mathcal{O}\Big(\frac{n^{\alpha+1}}{\log^3 n}\Big). \tag{27}$$

Adding (21), (26) and (27), we obtain finally

$$\lambda_{n+1}^{[1]} = [z^n]\tilde{\lambda}^{[1]}(z) = \frac{1}{\alpha+1}\frac{n^{\alpha+1}}{\log n} + \Big(\frac{1}{\alpha(\alpha+1)} + \frac{1}{(\alpha+1)^2} + \frac{\Psi(\alpha+1)}{\alpha+1}\Big)\frac{n^{\alpha+1}}{\log^2 n} + \mathcal{O}\Big(\frac{n^{\alpha+1}}{\log^3 n}\Big).$$

Substituting $n-1$ for n, it follows that expansion (4) is valid for $s=1$.

Now we assume that for a given $s \geq 2$, all moments $\mathbb{E}(Y_n^r) = \lambda_n^{[r]}$ with $1 \leq r < s$ have the expansion (4). To compute the asymptotic growth of the coefficients $[z^n]r^{[s]}(z)$, we use $[z^n]t^{\odot s_1}(z) = n^{s_1\alpha}$ and the following consequences of (16), (17) and (19), where the γ_r are given in Theorem 2.2:

$$[z^n]\Big(\frac{d}{dz}\lambda^{[s_2]}(z)\Big)\int_{t=0}^{z}\lambda^{[s_3]}(t)dt =$$

- $\dfrac{\Gamma\big(s_2(\alpha+1)+1\big)\Gamma\big(s_3(\alpha+1)-1\big)}{(\alpha+1)^{s_2+s_3}\Gamma\big((s_2+s_3)(\alpha+1)\big)}\dfrac{n^{(s_2+s_3)(\alpha+1)-1}}{\log^{s_2+s_3} n}\Big(1+\mathcal{O}\big(\dfrac{1}{\log n}\big)\Big)$, for $0 < s_2, s_3 < s$,
- $\dfrac{1}{(\alpha+1)^{s_3}(s_3(\alpha+1)-1)}\dfrac{n^{s_3(\alpha+1)-1}}{\log^{s_3} n}\Big(1+\mathcal{O}\big(\dfrac{1}{\log n}\big)\Big)$, for $s_2 = 0,\ 0 < s_3 < s$,
- $\dfrac{1}{(\alpha+1)^{s_2}}\dfrac{n^{s_2(\alpha+1)}}{\log^{s_2} n} + \gamma_{s_2}\dfrac{n^{s_2(\alpha+1)}}{\log^{s_2+1} n}\Big(1+\mathcal{O}\big(\dfrac{1}{\log n}\big)\Big)$, for $0 < s_2 < s,\ s_3 = 0$,
- $1+\mathcal{O}\big(\dfrac{1}{n}\big)$, for $s_2 = 0,\ s_3 = 0$.

Thus for $s_1+s_2+s_3 = s$ and the restrictions $0 \leq s_2, s_3 < s$, we can describe the growth of the summands of $[z^n]r^{[s]}(z)$:

$$[z^n]\binom{s}{s_1,\ s_2,\ s_3}t^{\odot s_1}\odot\Big[\Big(\frac{d}{dz}\lambda^{[s_2]}(z)\Big)\int_{t=0}^{z}\lambda^{[s_3]}(t)dt\Big] =$$

- $\dbinom{s}{s_1,\ s_2,\ s_3}\dfrac{\Gamma\big(s_2(\alpha+1)+1\big)\Gamma\big(s_3(\alpha+1)-1\big)}{(\alpha+1)^{s_2+s_3}\Gamma\big((s_2+s_3)(\alpha+1)\big)}\dfrac{n^{s\alpha+s_2+s_3-1}}{\log^{s_2+s_3} n}\Big(1+\mathcal{O}\big(\dfrac{1}{\log n}\big)\Big)$,
  for $0 < s_2, s_3 < s$,
- $\dbinom{s}{s_1}\dfrac{1}{(\alpha+1)^{s_3}(s_3(\alpha+1)-1)}\dfrac{n^{s\alpha+s_3-1}}{\log^{s_3} n}\Big(1+\mathcal{O}\big(\dfrac{1}{\log n}\big)\Big)$, for $s_2 = 0,\ 0 < s_3 < s$,
- $\dbinom{s}{s_1}\dfrac{1}{(\alpha+1)^{s_2}}\dfrac{n^{s\alpha+s_2}}{\log^{s_2} n} + \dbinom{s}{s_1}\gamma_{s_2}\dfrac{n^{s\alpha+s_2}}{\log^{s_2+1} n}\Big(1+\mathcal{O}\big(\dfrac{1}{\log n}\big)\Big)$, for $0 < s_2 < s,\ s_3 = 0$,
- $n^{s\alpha}\Big(1+\mathcal{O}\big(\dfrac{1}{n}\big)\Big)$, for $s_2 = 0,\ s_3 = 0$.

Therefore the maximum is obtained when choosing $s_3 = 0$ and $s_2 = s-1$, but additionally the instances $s_1 = 0$ and $0 < s_2, s_3 < s$ give contributions to the second order term. We get thus the expansion

$$[z^n]r^{[s]}(z) = \frac{s}{(\alpha+1)^{s-1}}\frac{n^{s(\alpha+1)-1}}{\log^{s-1} n} + \tilde{\gamma}_s\frac{n^{s(\alpha+1)-1}}{\log^{s} n} + \mathcal{O}\Big(\frac{n^{s(\alpha+1)-1}}{\log^{s+1} n}\Big),$$

with $\tilde{\gamma}_s := s\gamma_{s-1} + \sum_{j=1}^{s-1}\binom{s}{j}\frac{\Gamma\big(j(\alpha+1)+1\big)\Gamma\big((s-j)(\alpha+1)-1\big)}{(\alpha+1)^s\Gamma\big(s(\alpha+1)\big)}$. This gives by repeatedly applying (16) and (19) the following expansions:

$$[z^n]\frac{r^{[s]}(z)}{(1-z)L(z)} = \frac{1}{(\alpha+1)^s}\frac{n^{s(\alpha+1)}}{\log^s n} + \Big(\frac{\Psi(s(\alpha+1))}{(\alpha+1)^s} + \frac{1}{(\alpha+1)^{s+1}} + \frac{\tilde{\gamma}_s}{s(\alpha+1)}\Big)\frac{n^{s(\alpha+1)}}{\log^{s+1} n} + \mathcal{O}\Big(\frac{n^{s(\alpha+1)}}{\log^{s+2} n}\Big),$$

$$[z^n]\frac{1}{(1-z)^2L^2(z)}\int_{t=0}^{z}r^{[s]}(t)dt = \frac{1}{(\alpha+1)^s(s(\alpha+1)-1)}\frac{n^{s(\alpha+1)}}{\log^{s+1} n} + \mathcal{O}\Big(\frac{n^{s(\alpha+1)}}{\log^{s+2} n}\Big),$$

$$[z^n]\frac{2}{(1-z)^2}\int_{t=0}^{z}\frac{1}{(1-t)\log^3\frac{1}{1-t}}\int_{u=0}^{t} r^{[s]}(u)\,du\,dt = \mathcal{O}\Big(\frac{n^{s(\alpha+1)}}{\log^{s+2} n}\Big).$$

Combining these results and using (25) we obtain

$$\lambda_n^{[s]} = \frac{1}{(\alpha+1)^s}\frac{n^{s(\alpha+1)}}{\log^s n} + \gamma_s\frac{n^{s(\alpha+1)}}{\log^{s+1} n} + \mathcal{O}\Big(\frac{n^{s(\alpha+1)}}{\log^{s+2} n}\Big),$$

where γ_s satisfies for $s \geq 1$ the following recurrence with inital value $\gamma_0 := 0$:

$$\gamma_s = \frac{\gamma_{s-1}}{\alpha+1} + \frac{\Psi\big(s(\alpha+1)\big)}{(\alpha+1)^s} + \frac{1}{(\alpha+1)^{s+1}} + \frac{1}{(\alpha+1)^s(s(\alpha+1)-1)} + \frac{1}{(\alpha+1)^s}\sum_{j=1}^{s-1}\binom{s}{j}\frac{\Gamma\big(j(\alpha+1)+1\big)\Gamma\big((s-j)(\alpha+1)-1\big)}{\Gamma\big(s(\alpha+1)+1\big)}.$$

Solving this recurrence leads exactly to the expression for γ_s given in Theorem 2.2. Analogous computations as in Section 5 leads from (4) to the stated result (5) for the centered moments.

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