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CUTTING DOWN VERY SIMPLE TREES

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Abstract. We study here, by using a recursive approach, the number of random cuts that are necessary to destroy a random tree of size n for simply generated tree families. Crucial for the applicability of such a recursive approach is a "randomness-preservation" property when cutting off a random edge. We can fully characterize the subclass of simply generated tree families, which satisfy this property and show then for for all these tree families that the number of random cuts to destroy a random size-n tree is asymptotically, for $n \to \infty$, Rayleigh distributed.

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1. Introduction. In this paper we consider the following *cutting down* procedure to "destroy" trees. We start with a tree T of size $|T| = n$, where the size measures as usual the number of nodes of T. If $n \geq 2$, then we choose one of the $n-1$ edges in the tree and remove this edge from T. After removing this edge, the original tree T falls into two subtrees, where one of them, which we will denote by T', contains the root and has size k, with $1 \leq k \leq n-1$. If $k \geq 2$, this edge-removing procedure will be iterated with T': we choose one of the $k-1$ edges of T' and remove it, take the subtree containing the root, and so on. After $m \leq n-1$ steps we will obtain a tree consisting only of the root itself and we stop. An example for cutting down a tree is given in Figure 1.

The aim of this work is to study for certain tree families, how many edges will be removed by the *randomized* cutting down procedure of a *randomly* chosen tree of size n until the root is isolated or equivalently, how many steps are done, until the tree is destroyed. The used probability models are described next.

All tree families considered here are *simply generated tree families.* As it is described in [1], a class $\mathcal{T}$ of simply generated trees can be defined by the following way. A sequence of non-negative numbers $(\varphi_i)_{i\geq 0}$ with $\varphi_0 > 0$ is used to define the weight $w(T)$ of any planted plane tree (= ordered tree) T by $w(T) = \prod_v \varphi_{d(v)}$, where v ranges over all vertices of T and $d(v)$ is the out-degree (the number of children) of v. The family $\mathcal{T}$ consists then of all trees T together with its weights $w(T)$. It follows further that the generating function $T(z) = \sum_{n\geq 0} T_n z^n$ of the quantity $T_n := \sum_{|T|=n} w(T)$ satisfies the (formal) functional equation

$$T(z) = z\varphi\big(T(z)\big), \tag{1}$$

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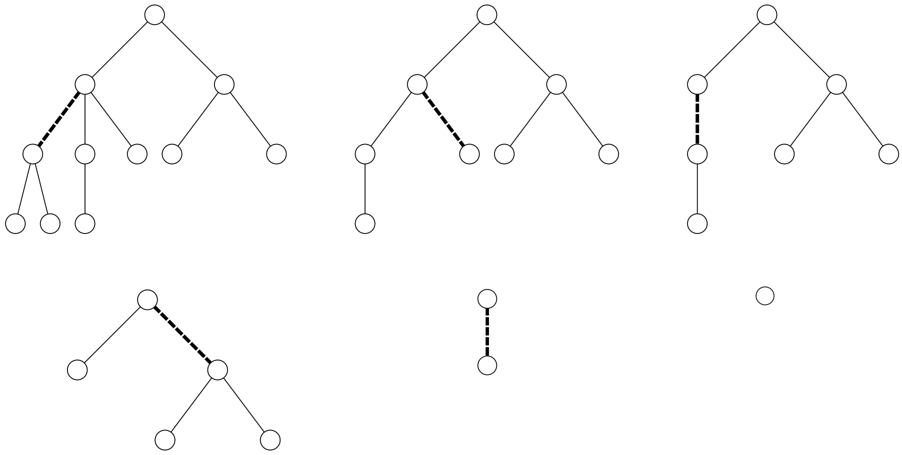

Figure 1: Destroying a tree of size 11 in 5 steps, where the dotted lines mark the edges, which will be removed in the next step.

where the degree-weight generating function $\varphi(t)$ is given as formal power series $\varphi(t) = \sum_{i \geq 0} \varphi_i t^i$. If all φ_i are nonnegative integers, then T_n counts the number of trees in $\mathcal{T}$ with size n.

Combinatorially, a simply generated tree family $\mathcal{T}$ can be described by the formal recursive equation

$$\mathcal{T} = \circ \times \varphi(\mathcal{T}), \tag{2}$$

with $\circ$ a node and $\varphi(\mathcal{T})$ the substituted structure (see, e.g., [6]).

As the model of randomness we will always assume here the random tree model and we speak about *random* simply generated trees. This means, that we choose a tree of size n with probability proportional to its weight, if there exists a tree of size n with positive weight, otherwise the probability is zero.

Further, we will always assume for the cutting down procedure that the removed edges are at each stage chosen at random from the remaining tree containing the root. We speak thus about the *randomized* cutting down procedure.

Summarizing, we will study here for simply generated tree families $\mathcal{T}$ the random variable X_n, which counts the number of edges that will be removed from a random simply generated tree of size n by the randomized cutting down procedure, until the root is isolated.

This problem was studied first by Meir and Moon in [9] for random labelled unordered trees, so called Cayley trees, which can be considered as a special instance of a simply generated tree family. Using a recursive approach the first two moments of X_n were computed exactly and asymptotically for Cayley trees. The present work was motivated by characterizing the limiting distribution of X_n for Cayley trees and extending the results to other instances of simply generated tree families by using the recursive approach introduced in [9].

It turns out that the following "randomness-preservation" property of a tree family $\mathcal{T}$ is crucial for the applicability of such a recursive approach, where we denote by $\mathcal{T}_n$ the set of trees in $\mathcal{T}$ with n nodes.

Randomness-preservation property: Choose a random simply generated tree from the family $\mathcal{T}_n$ and then one of its $n-1$ edges uniformly at random. Cutting this edge produces a pair of trees of size k (the one that contains the root) and $n-k$. Then the subtrees themselves are random simply generated trees from the family $\mathcal{T}_k$ and $\mathcal{T}_{n-k}$.

It is a simple consequence of the recursive description of simply generated tree families that randomness is always preserved for the subtree which does not contain the original root, whereas it turns out that, in general, the subtree containing the original root is not a random tree of $\mathcal{T}_k$ anymore. On the other hand several important simply generated tree families (e. g., d-ary trees, planted plane trees, and of course Cayley trees) have this randomness-preservation property. Throughout this paper simply generated tree families satisfying the randomness-preservation property are called *very simple tree families*.

A study of this property leading to a full characterization of very simple tree families by their degree-weight generating function is done in the present paper and the corresponding result is given by the following lemma.

LEMMA 1. *A simply generated tree family satisfies the randomness-preservation property and is thus a very simple tree family if and only if the degree-weight generating function $\varphi(t) = \sum_{i\geq 0} \varphi_i t^i$ is given by one of the following three formulæ.*

$$\begin{aligned}
&\text{Case A}: \quad \varphi(t) = \varphi_0 e^{\alpha_0 t},\\
&\text{Case B}: \quad \varphi(t) = \varphi_0 \left(1 + \frac{\alpha_0 t}{d}\right)^d, \;\; d \geq 2,\\
&\text{Case C}: \quad \varphi(t) = \frac{\varphi_0}{(1-(2\alpha_1-\alpha_0)t)^{\frac{\alpha_0}{2\alpha_1-\alpha_0}}},
\end{aligned}$$

with $\varphi_0 > 0$, $\alpha_0 > 0$, $2\alpha_1 - \alpha_0 > 0$. □

As it is illustrated by Figure 2, randomness is preserved by cutting off a random edge for random planted plane trees, but it is not preserved for random Motzkin trees.

If we denote as usual by $\xrightarrow{(d)}$ the convergence in distribution, we can formulate our main result:

THEOREM 2. *The number of random cuts X_n, that will be done to destroy a random chosen tree of size n from a very simple tree family with a degree-weight generating function $\varphi(t)$ given by Lemma 1, is asymptotically for $n \to \infty$ Rayleigh distributed with parameter $\sqrt{\frac{\tau\varphi''(\tau)}{\varphi'(\tau)}}$:*

$$\frac{X_n}{\sqrt{n}} \xrightarrow{(d)} X,$$

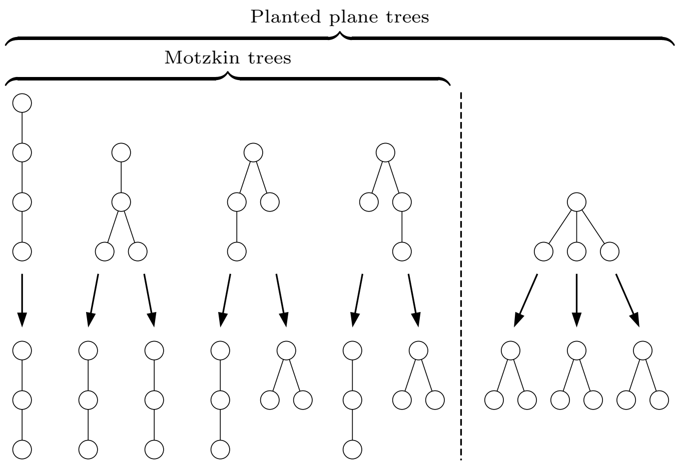

Figure 2: All possibilities of choosing a Motzkin tree ($\varphi(t) = 1 + t + t^2$) resp. a planted plane tree ($\varphi(t) = \frac{1}{1-t}$) of size 4, where by cutting off an edge, the remaining subtree containing the root is of size 3. The two possible types of trees of size 3 will appear as remaining subtrees with equal probability for planted plane trees, whereas the chain will appear more often for Motzkin trees.

where X is a random variable with density function

$$f(x) = \frac{\varphi'(\tau)x}{\tau\varphi''(\tau)} e^{-\frac{\varphi'(\tau)x^2}{2\tau\varphi''(\tau)}}, \quad \text{for } x \geq 0, \quad \text{and} \quad f(x) = 0 \text{ otherwise},$$

and where τ is given by the unique solution of $\varphi(t) = t\varphi'(t)$ of smallest modulus.

Moreover, the r-th moments $\mathbb{E}(X_n^r)$ are asymptotically for $n \to \infty$ given by

$$\mathbb{E}(X_n^r) = \sum_{m \geq 0} m^r \mathbb{P}\{X_n = m\} \sim \left(\frac{2\tau\varphi''(\tau)}{\varphi'(\tau)}\right)^{\frac{r}{2}} \Gamma\left(\frac{r}{2} + 1\right) n^{\frac{r}{2}}. \qquad \square$$

The Rayleigh distribution appears also as limit law of other functionals in random simply generated tree families, e.g., as limiting distribution of the depth (the number of ancestors) of a randomly chosen node. If $X_{[\beta]}$ denotes a random variable that is Rayleigh distributed with parameter β, it has the density function

$$f_{[\beta]}(x) = \frac{x}{\beta^2} e^{-\frac{1}{2}(\frac{x}{\beta})^2}, \quad \text{for } x \geq 0, \quad \text{and} \quad f_{[\beta]}(x) = 0 \text{ otherwise}.$$

The r-th moments are given by

$$\mathbb{E}(X_{[\beta]}^r) = 2^{\frac{r}{2}} \beta^r \Gamma\left(\frac{r}{2} + 1\right).$$

Remark. It has to be remarked that Janson [7] has also recently studied the cutting down procedure, where he has revealed a very useful connection to the

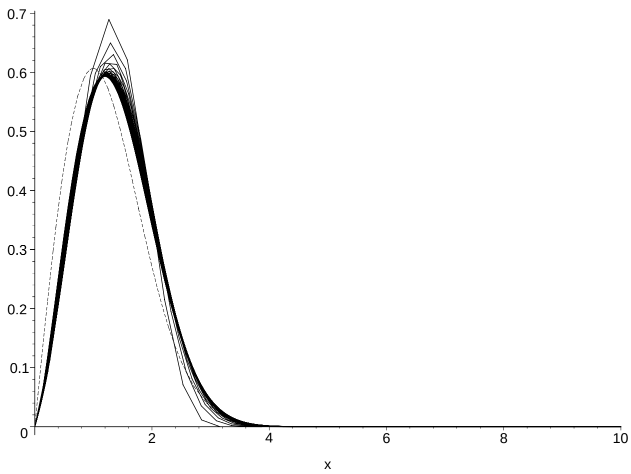

Figure 3: Comparison of the probability functions of the (normalized) random variables $X_{10}, X_{15}, \ldots, X_{200}$ and (in dotted lines) the density function of the limiting distribution X for random Cayley trees.

number of records in the tree when edges are assigned random labels. Using this connection together with Aldous' theory of the continuum random tree [1] it was shown in [7] that X_n is indeed asymptotically Rayleigh distributed for (essentially) all simply generated tree families, i. e., also for those simply generated tree families that do not satisfy the randomness-preservation property. However, the approach presented here might be also of interest due to the following reasons.

1. The present approach leads with relatively little effort, i. e., without requiring relations between simply generated trees and the Brownian excursion, to a limiting distribution result of X_n for several important instances of simply generated tree families. On the other hand, as a consequence of the description via the number of records together with the theory of the continuum random tree, an "invariance principle" for the parameter studied holds. Suitably scaled, all simply generated tree families must have the same limiting distribution and thus it is actually enough to identify the limiting distribution for one particular simply generated tree family (or several instances as carried out here) to show the general result (see [7]).

2. As shown in [3] the present approach can be extended naturally to study the "total cost" of cutting down trees when cutting an edge costs a certain toll depending on the size of the tree. Such questions are of interest, e.g., when studying certain probabilistic models involved in the Union-Find algorithm. It seems that for such questions the approach used in [7] cannot be applied easily.

3. It turns out that the randomness-preservation property and in particular the characterization of very simple tree families as given here is of importance

in its own: it has been shown in [10] that it is equivalent to the characterization of the subclass of *increasing tree families*, which can be described via a tree evolution process. Increasing tree families can be considered as tree families generated from simply generated tree families by equipping the trees with increasing labellings, i. e., the nodes of a tree of size n are labelled by distinct integers of the set $\{1, \dots, n\}$ in such a way that each sequence of labels along any path starting at the root is increasing (see [2]). Important instances of increasing tree families are, e. g, recursive trees, plane-oriented recursive trees and binary increasing trees, which are of interest for various applications (see [2, 8]). The characterization of increasing tree families that can be described via a tree evolution process as given in [10] relies heavily on the considerations of the present paper, i. e., on the detailed study of the randomness-preservation property as carried out in Section 3. The property that a certain tree family can be constructed via a tree evolution process often turns out to be important in the analysis of parameters for random trees, e.g. it is then a priori clear for such a tree family that the expected height and the expected width of trees of size n are monotone functions (see also remarks on corresponding problems for simply generated tree families in [7]).

2. Outline of the method of proof.

2.1. Mathematical preliminaries. In our study of the parameter X_n considered for very simple tree families, we will use a generating functions approach and translate the first established recurrence (4) for the probabilities $\mathbb{P}\{X_n = m\}$ into differential equations for suitably defined generating functions. We are able to give closed form solutions for these differential equations. In order to extract coefficients from the generating functions asymptotically, we will expand around the dominant singularity (for the tree families considered here the dominant singularity is uniquely determined as the unique dominant singularity of the generating function $T(z)$) and apply singularity analysis (see [5]).

Therefore we give here some well known results about $T(z)$, as defined via equation (1) by the degree-weight generating function $\varphi(t)$, that are required in our analysis. It follows from the implicit function theorem that the equation $z = \frac{t}{\varphi(t)}$ is not invertible at solutions of $\varphi(t) = t\varphi'(t)$. If one supposes that $\varphi(t)$ has a positive radius of convergence $R > 0$ and one further assumes, that there exists a minimal positive solution $\tau < R$ of the equation $t\varphi'(t) = \varphi(t)$, then the number p of solutions of smallest modulus is given by the "period" $p := \gcd\{i \,|\, \varphi_i > 0\}$. For all very simple tree families, it holds $p = 1$ and thus, that τ is the only solution of smallest modulus, and it follows then further, that $T(z)$ has only one dominant singularity $z = \rho$ with $\rho = \frac{\tau}{\varphi(\tau)} = \frac{1}{\varphi'(\tau)}$. Expanding $T(z)$ around the singularity $z = \rho$ gives

$$T(z) = \tau - \sqrt{\frac{2\tau}{\rho\varphi''(\tau)}}\sqrt{1 - \frac{z}{\rho}} + \mathcal{O}\left(1 - \frac{z}{\rho}\right),$$

and via singularity analysis also the asymptotic expansion of the coefficients

$$T_n = \frac{\sqrt{\varphi(\tau)}}{\sqrt{2\pi\varphi''(\tau)}\,\rho^n\, n^{\frac{3}{2}}} \left(1 + \mathcal{O}\Big(\frac{1}{n}\Big)\right). \tag{3}$$

2.2. The recursive approach. We will study the random variable X_n by treating the recurrence

$$\mathbb{P}\{X_n = m\} = \sum_{k=1}^{n-1} p_{n,k}\, \mathbb{P}\{X_k = m-1\}, \quad \text{for } n \geq 2,\ m \geq 1, \tag{4}$$

with initial values $\mathbb{P}\{X_1 = 0\} = 1$ and $\mathbb{P}\{X_n = 0\} = 0$ for $n \geq 2$. Here $p_{n,k}$ denotes the probability that, by choosing a random tree of size n from the given tree family and removing a random edge, the remaining subtree containing the root is of size k. This approach was used in [9] to compute the first two moments of X_n for Cayley trees.

In order to apply this approach for a certain tree family it is necessary that the randomness-preservation property as formulated in Section 1 is satisfied. This subclass of simply generated tree families, which is called here the class of *very simple tree families*, is quite small, but it contains very prominent members. The proof of Lemma 1 and thus of the characterization of very simple tree families by the degree-weight generating function $\varphi(t)$ is given in Section 3.

For all very simple tree families, the distributional recurrence (4) holds for X_n. It remains to determine the probabilities $p_{n,k}$, which is carried out in Section 4. It turns out that one can even give explicit formulæ for the quantities $p_{n,k}$.

Finally, to solve the recurrence, we will use a generating functions approach and define the bivariate generating function

$$M(z,v) = \sum_{n\geq 1} \sum_{0\leq m\leq n} T_n\, \mathbb{P}\{X_n = m\} z^n v^m. \tag{5}$$

Due to the "nice structure" of the probabilities $p_{n,k}$, which are given as formulæ (14), (15) and (16), we can translate (4) into linear first order differential equations for the generating functions $M(z,v)$. Moreover, these differential equations can be solved exactly. To establish the limiting distribution result for X_n, we will use the so called method of moments. By expanding $M(z,v)$ around $v = 1$ and extracting the coefficients of z^n asymptotically via singularity analysis, we will give an asymptotic equivalent for every r-th moment $\mathbb{E}(X_n^r)$. It turns out that, after a suitable normalization, these moments converge to the moments of the Rayleigh distribution. Since the Rayleigh distribution is fully characterized by its moments, an application of the theorem of Fréchet and Shohat (see, e.g., [4]) shows that the limiting distribution result, Theorem 2, holds. This is worked out in Section 5.

3. Characterization of very simple trees. Here we want to describe in detail, which simply generated tree families satisfy the randomness-preservation property. If we denote by $T \xrightarrow{e} T'$ the property, that by removing edge e of T, the remaining subtree of T containing the root is T', we can formalize the required property by the following two conditions, which must both be satisfied:

$$(i) \qquad \frac{w(T')}{w(T'')} = \frac{\sum\limits_{(T,e):|T|=n, T \stackrel{e}{\longrightarrow} T'} w(T)}{\sum\limits_{(T,e):|T|=n, T \stackrel{e}{\longrightarrow} T''} w(T)},$$

for all $T', T'' : \ w(T') \neq 0, w(T'') \neq 0, |T'| = |T''| = k$, with $1 \leq k \leq n-1$ and all $n \geq 2$,

$$(ii) \qquad T \stackrel{e}{\to} T' \text{ and } w(T') = 0 \quad \Rightarrow \quad w(T) = 0 \quad \text{for all } T, T'.$$

If $w(T') = 0$ then it follows by definition that there exists a j, such that $\varphi_j = 0$. By choosing for T' the star with $j+1$ nodes, consisting of a root with j sons (thus $w(T') = \varphi_0^j \varphi_j = 0$), and for T the star with $j+2$ nodes (thus $w(T) = \varphi_0^{j+1}\varphi_{j+1}$) and for e an arbitrary edge of T, condition (ii) implies, that $\varphi_{j+1} = 0$ must also hold. By iterating this argument, we get, that $\varphi_i = 0$ must hold for all $i \geq j$. We can choose $d := j-1$ minimal, such that also $\varphi_i > 0$ for all $0 \leq i \leq j-1 = d$, and get thus for (ii) an equivalent condition.

(ii') If there exists a j, such that $\varphi_j = 0$, then there exists a node degree $d < j$, such that $\varphi_i = 0$ for all $i > d$ and $\varphi_i > 0$ for all $0 \leq i \leq d$.

Searching for all (T, e) with $|T| = n$ such that $T \stackrel{e}{\to} T'$ for a given T' with $|T'| = k$ is of course the same as searching for all possibilities of adding an arbitrary tree $\tilde{T}$ with $|\tilde{T}| = n-k$ to one of the k nodes of T'. If we select a node v of T' with out-degree $d(v)$, then we have $d(v)+1$ possibilities of appending $\tilde{T}$ at v and the out-degree of v increases by one. This gives for (i) the equivalent condition

$$\frac{\prod_{v \in T'} \varphi_{d(v)}}{\prod_{v \in T''} \varphi_{d(v)}} = \frac{\left(\sum_{\tilde{T}:|\tilde{T}|=n-k} \prod_{v \in \tilde{T}} \varphi_{d(v)}\right) \prod_{v \in T'} \varphi_{d(v)} \sum_{v \in T'} \frac{(d(v)+1)\varphi_{d(v)+1}}{\varphi_{d(v)}}}{\left(\sum_{\tilde{T}:|\tilde{T}|=n-k} \prod_{v \in \tilde{T}} \varphi_{d(v)}\right) \prod_{v \in T''} \varphi_{d(v)} \sum_{v \in T''} \frac{(d(v)+1)\varphi_{d(v)+1}}{\varphi_{d(v)}}},$$

for all $T', T'' : \ w(T') \neq 0, w(T'') \neq 0, |T'| = |T''| = k$, with $k \geq 1$ and $n - k \geq 1$, which simplifies to

$$\sum_{v \in T'} \frac{(d(v)+1)\varphi_{d(v)+1}}{\varphi_{d(v)}} = \sum_{v \in T''} \frac{(d(v)+1)\varphi_{d(v)+1}}{\varphi_{d(v)}},$$

for all $T', T'' : \ w(T') \neq 0, w(T'') \neq 0, |T'| = |T''| = k \geq 1$.

Thus we have to find all degree-weight sequences $(\varphi_i)_{i \geq 0}$, such that

$$\sum_{v \in T} \frac{(d(v)+1)\varphi_{d(v)+1}}{\varphi_{d(v)}}$$

is for all trees T with $w(T) \neq 0$ depending only on the size $|T|$ of T.

We will use throughout this paper the abbreviation $\alpha_i := \frac{\varphi_{i+1}}{\varphi_i}$, $i \geq 0$. Further we denote for a tree T by k_i, $i \geq 0$ the number of nodes of out-degree i in T. If $\varphi_i > 0$ for all i, then the α_i are defined for all i, which implies $w(T) \neq 0$ for all trees T and condition (ii) is superfluous. But if the node degrees are bounded by d, thus $w(T) = 0$ for every tree T, where a $j > d$ exists with $k_j > 0$, we will have by (ii') the restriction that for condition (i) only those trees are considered, where $k_i = 0$ for all $i > d$. In order to avoid treating both cases separately, we set in the next formula the upper summation bound d for the latter case in brackets.

Using the relations for the number of nodes resp. the number of edges in a tree T, we can reformulate condition (i) as the following system of equations

$$(i') \quad \sum_{i=0}^{\infty\,(d)} (i+1)\alpha_i k_i = c(k), \quad \sum_{i=0}^{\infty\,(d)} k_i = k, \quad \sum_{i=0}^{\infty\,(d)} i k_i = k-1,$$

where $c(k)$ is a function defined for the positive integers. Eliminating k_0 resp. k_1 by using the node-sum equation resp. the edge-sum equation, we obtain the equivalent condition

$$\sum_{i=2}^{\infty\,(d)} \big[(i+1)\alpha_i - 2i\alpha_1 + (i-1)\alpha_0\big] k_i = \tilde{c}(k), \quad \sum_{i=0}^{\infty\,(d)} k_i = k, \quad \sum_{i=0}^{\infty\,(d)} i k_i = k-1, \tag{6}$$

for a function $\tilde{c}(k)$.

To satisfy (6), the following condition is of course sufficient:

$$(i'') \quad \beta_i := (i+1)\alpha_i - 2i\alpha_1 + (i-1)\alpha_0 = 0, \quad \text{for all } 2 \le i \ \ (\, 2 \le i \le d).$$

To show, that condition (i'') is also necessary to satisfy (6) resp (i'), we assume that $\beta_i = 0$ does not hold for all $2 \le i\,(\le d)$. Then there exists a $j \ge 2$, such that $\beta_j \ne 0$, but $\beta_i = 0$ for $2 \le i < j\,(\le d)$. We choose then for T' the chain with $j+1$ nodes: $k_0 = 1$, $k_1 = j$ and $k_i = 0$, for $i \ge 2$. For T'' we choose the star with $j+1$ nodes: $k_0 = j$, $k_j = 1$, $k_i = 0$ otherwise. But we get then that (6) is not satisfied: for T' we obtain $\sum_{i\ge 2} \beta_i k_i = 0$, whereas for T'' we get $\sum_{i \ge 2} \beta_i k_i = \beta_j \ne 0$.

Thus it remains to study (i'') and we obtain

$$\alpha_i = \frac{2i\alpha_1 - (i-1)\alpha_0}{i+1}, \quad \text{for } i \ge 0. \tag{7}$$

By using

$$\varphi_i = \varphi_0 \prod_{j=0}^{i-1} \alpha_j \quad \text{and} \quad \alpha_i = \frac{1}{\varphi_0\varphi_1} \frac{2i\varphi_0\varphi_2 - (i-1)\varphi_1^2}{i+1}, \quad \text{for } i \ge 0,$$

we obtain

$$\varphi_i = \varphi_0^{-i+1} \varphi_1^{-i} \frac{1}{i!} \prod_{j=0}^{i-1} \Big(j(2\varphi_0\varphi_2 - \varphi_1^2) + \varphi_1^2 \Big). \tag{8}$$

We distinguish now, whether $2\varphi_0\varphi_2 - \varphi_1^2 = 0$ or not.

- CASE A. If $2\varphi_0\varphi_2 - \varphi_1^2 = 0$ or equivalently $2\alpha_1 - \alpha_0 = 0$, we obtain for $i \geq 0$:

$$\varphi_i = \frac{\varphi_0^{-i+1}\varphi_1^i}{i!},$$

and thus

$$\varphi(t) = \sum_{i\geq 0} \varphi_i t^i = \varphi_0 \sum_{i\geq 0} \frac{(\varphi_0^{-1}\varphi_1 t)^i}{i!} = \varphi_0 e^{\alpha_0 t}, \tag{9}$$

with $\varphi_0 > 0$ and $\alpha_0 = \frac{\varphi_1}{\varphi_0} > 0$.

If $2\varphi_0\varphi_2 - \varphi_1^2 \neq 0$, we proceed from (8) with

$$\varphi_i = \varphi_0^{-i+1}\varphi_1^{-i}(2\varphi_0\varphi_2 - \varphi_1^2)^i \frac{1}{i!} \prod_{j=0}^{i-1} \left(j + \frac{\varphi_1^2}{2\varphi_0\varphi_2 - \varphi_1^2} \right). \tag{10}$$

- CASE B. Next we treat the case, where a $d \geq 0$ exists, such that $d + \frac{\varphi_1^2}{2\varphi_0\varphi_2 - \varphi_1^2} = 0$. This is equivalent to $2d\alpha_1 = (d-1)\alpha_0$ and via (7) thus to $\alpha_d = 0$. Therefore, this case treats exactly the instances, where the degree is bounded by d: $\varphi_i = 0$ for $i > d$ and $\varphi_i > 0$ for $0 \leq i \leq d$. We obtain from (7) further, that

$$\alpha_i = \frac{2i\frac{d-1}{2d}\alpha_0 - (i-1)\alpha_0}{i+1} = \frac{d-i}{d(i+1)}\alpha_0,$$

for $0 \leq i \leq d-1$. This gives for $0 \leq i \leq d$:

$$\varphi_i = \varphi_0 \prod_{j=0}^{i-1} \left(\frac{d-j}{d(j+1)}\alpha_0 \right) = \frac{\varphi_0^{-i+1}\varphi_1^i}{d^i} \binom{d}{i}.$$

Thus the degree-weight generating function is given by

$$\varphi(t) = \sum_{i=0}^{d} \varphi_i t^i = \varphi_0 \sum_{i=0}^{d} \binom{d}{i} \left(\frac{\varphi_1 t}{d\varphi_0} \right)^i = \varphi_0 \left(1 + \frac{\alpha_0 t}{d} \right)^d, \tag{11}$$

with $\varphi_0 > 0$, $\alpha_0 > 0$ and where we demand $d \geq 2$, since we want to exclude the degenerate cases $(d = 0, d = 1)$, where all trees with positive weights are chains.

- CASE C. It remains to consider the case, where $j + \frac{\varphi_1^2}{2\varphi_0\varphi_2 - \varphi_1^2} \neq 0$ for all j. We obtain from (10) for $i \geq 0$:

$$\varphi_i = \varphi_0^{-i+1}\varphi_1^{-i}(2\varphi_0\varphi_2 - \varphi_1^2)^i \binom{i - 1 + \frac{\varphi_1^2}{2\varphi_0\varphi_2 - \varphi_1^2}}{i}.$$

The degree-weight generating function is thus given by

$$\begin{aligned}\varphi(t) &= \sum_{i\ge 0}\varphi_i t^i = \varphi_0 \sum_{i\ge 0}\binom{i-1+\frac{\varphi_1^2}{2\varphi_0\varphi_2-\varphi_1^2}}{i}\Big(\varphi_0^{-1}\varphi_1^{-1}(2\varphi_0\varphi_2-\varphi_1^2)t\Big)^i \\ &= \frac{\varphi_0}{(1-(2\alpha_1-\alpha_0)t)^{\frac{\alpha_0}{2\alpha_1-\alpha_0}}}, \qquad (12)\end{aligned}$$

with $\varphi_0 > 0$, $\alpha_0 > 0$ and $2\alpha_1 - \alpha_0 > 0$.

This completes the proof of Lemma 1.

4. The transition probabilities. Here we want to compute for very simple tree families the transition probabilities $p_{n,k}$ as defined by (4). First we will study these quantities for general simply generated tree families and introduce for $T(z) = \sum_{n\ge 1} T_n z^n$ with degree-weight generating function $\varphi(t)$ the parameters $G_{n,m}$ and $H_{n,m}$.

$G_{n,m}$ denotes the probability, that the number of descendants of a randomly chosen node (including the node itself) in a random tree of size n is m.

$H_{n,m}$ denotes the probability, that the number of descendants of a node, which was chosen at random from one of the $n-1$ non-root nodes in a random tree of size n, is m. We obtain then of course the relation $p_{n,k} = H_{n,n-k}$.

Introducing the generating functions

$$G(z,v) = \sum_{n\ge 1}\sum_{0\le m\le n} nT_n\, G_{n,m}\, z^n v^m,$$
$$H(z,v) = \sum_{n\ge 1}\sum_{0\le m\le n} (n-1)T_n\, H_{n,m}\, z^n v^m,$$

we obtain by translating the formal equation (2) the equations

$$G(z,v) = T(zv) + \sum_{i\ge 1} i\varphi_i z G(z,v) T(z)^{i-1} = T(zv) + z\varphi'\big(T(z)\big)G(z,v),$$
$$H(z,v) = \sum_{i\ge 1} i\varphi_i z G(z,v) T(z)^{i-1} = z\varphi'\big(T(z)\big)G(z,v),$$

which implies

$$G(z,v) = \frac{T(zv)}{1 - z\varphi'\big(T(z)\big)}, \quad \text{and}$$
$$H(z,v) = \frac{z\varphi'(T(z))T(zv)}{1 - z\varphi'\big(T(z)\big)} = T(zv)\left(\frac{1}{1 - z\varphi'\big(T(z)\big)} - 1\right).$$

To extract coefficients, we use Cauchy's integral formula and get

$$[z^n]\frac{1}{1-z\varphi'\big(T(z)\big)} = \frac{1}{2\pi i}\int \frac{1}{z^{n+1}}\frac{1}{1-z\varphi'\big(T(z)\big)}dz = \\ = \frac{1}{2\pi i}\int \frac{\big(\varphi(T)\big)^n}{T^{n+1}}dT = [T^n]\big(\varphi(T)\big)^n,$$

which leads for $1 \le m \le n-1$ to

$$[z^n v^m]H(z,v) = [z^m v^m]T(zv)[z^{n-m}]\left(\frac{1}{1-z\varphi'\big(T(z)\big)} - 1\right) = \\ = T_m\,[T^{n-m}]\big(\varphi(T)\big)^{n-m}.$$

Thus we find, that the transition probabilities $p_{n,k}$ are for simply generated trees given by

$$p_{n,k} = \frac{T_{n-k}\,[T^k]\big(\varphi(T)\big)^k}{(n-1)T_n}, \quad \text{for } 1 \le k \le n-1. \tag{13}$$

We will specialize now for $\varphi(t)$ to obtain the results for very simple tree families. We also use for computing T_k the Lagrange inversion formula (see, e.g., [11]):

$$T_k = [z^k]T(z) = \frac{1}{k}[T^{k-1}]\big(\varphi(T)\big)^k, \quad k \ge 1.$$

- Case A. $\varphi(t) = \varphi_0 e^{\alpha_0 t}$. We obtain

$$T_k = \frac{1}{k}[T^{k-1}]\varphi_0^k e^{\alpha_0 kT} = \varphi_0^k\frac{\alpha_0^{k-1}k^{k-1}}{k!},$$

  and thus

$$[T^k]\big(\varphi(T)\big)^k = [T^k]\varphi_0^k e^{\alpha_0 kT} = \varphi_0^k\frac{\alpha_0^k k^k}{k!} = \alpha_0 k T_k.$$

  Here, the probabilities $p_{n,k}$ are via (13) thus given by

$$p_{n,k} = \frac{\alpha_0 k\, T_k T_{n-k}}{(n-1)\,T_n}. \tag{14}$$

- Case B. $\varphi(t) = \varphi_0\left(1+\frac{\alpha_0 t}{d}\right)^d$. This gives

$$T_k = \frac{1}{k}[T^{k-1}]\varphi_0^k\left(1+\frac{\alpha_0 T}{d}\right)^{kd} = \frac{\varphi_0^k}{k}\binom{kd}{k-1}\frac{\alpha_0^{k-1}}{d^{k-1}},$$

  and further

$$[T^k]\big(\varphi(T)\big)^k = [T^k]\varphi_0^k\left(1+\frac{\alpha_0 T}{d}\right)^{kd} = \varphi_0^k\binom{kd}{k}\frac{\alpha_0^k}{d^k} = (k(d-1)+1)\frac{\alpha_0}{d}T_k.$$

  The probabilities $p_{n,k}$ are therefore given by

$$p_{n,k} = \frac{\alpha_0(k(d-1)+1)\,T_k T_{n-k}}{d(n-1)\,T_n}. \tag{15}$$

- CASE C. $\varphi(t) = \frac{\varphi_0}{(1-(2\alpha_1-\alpha_0)t)^{\frac{\alpha_0}{2\alpha_1-\alpha_0}}}$. This leads to

$$T_k = \frac{1}{k}[T^{k-1}]\frac{\varphi_0^k}{(1-(2\alpha_1-\alpha_0)t)^{\frac{\alpha_0 k}{2\alpha_1-\alpha_0}}} = \frac{\varphi_0^k}{k}\binom{\frac{\alpha_0 k}{2\alpha_1-\alpha_0}+k-2}{k-1}(2\alpha_1-\alpha_0)^{k-1},$$

and

$$\begin{aligned}[T^k]\big(\varphi(t)\big)^k &= [T^k]\frac{\varphi_0^k}{(1-(2\alpha_1-\alpha_0)t)^{\frac{\alpha_0 k}{2\alpha_1-\alpha_0}}}\\ &= \varphi_0^k\binom{\frac{\alpha_0 k}{2\alpha_1-\alpha_0}+k-1}{k}(2\alpha_1-\alpha_0)^k\\ &= (2\alpha_1 k - 2\alpha_1+\alpha_0)T_k.\end{aligned}$$

The probabilities $p_{n,k}$ are determined by

$$p_{n,k} = \frac{(2\alpha_1 k - 2\alpha_1 + \alpha_0)\, T_k T_{n-k}}{(n-1)\, T_n}. \tag{16}$$

5. Solving the recurrences. As described in Section 2 the basic idea in our approach is the study of the recurrence (4), where the transition probabilities $p_{n,k}$ are computed in Section 4.

In the following we will figure out this analysis only for CASE C, since the way of proving Theorem 2 is for CASE A and CASE B fully analogous.

With $M(z,v)$ as defined by (5), we translate the recurrence

$$\mathbb{P}\{X_n = m\} = \sum_{k=1}^{n-1}\frac{(2\alpha_1 k - 2\alpha_1+\alpha_0)\, T_k T_{n-k}}{(n-1)\, T_n}\,\mathbb{P}\{X_k = m-1\}, \quad \text{for } n\geq 2,\ 1\leq m\leq n,$$

with $\mathbb{P}\{X_1 = 0\} = 1$ and $\mathbb{P}\{X_n = 0\} = 0$ for $n \geq 2$ into the following linear first order differential equation

$$z\frac{\partial}{\partial z}M(z,v) - M(z,v) = 2\alpha_1 v z T(z)\frac{\partial}{\partial z}M(z,v) - (2\alpha_1-\alpha_0)vT(z)M(z,v), \quad M(0,v) = 0,$$

or equivalently into

$$\frac{\frac{\partial}{\partial z}M(z,v)}{M(z,v)} = \frac{1-(2\alpha_1-\alpha_0)vT(z)}{z(1-2\alpha_1 vT(z))}. \tag{17}$$

To integrate the right side of (17), we change variables by using

$$dt = \frac{1 - 2\alpha_1 T}{\varphi(T)(1 - (2\alpha_1 - \alpha_0)T)} dT,$$

which follows of course immediately from (1) and the definition of $\varphi(t)$, and obtain

$$\begin{aligned}
&\log(M(z,v)) = \\
&= \int^z \frac{1 - (2\alpha_1 - \alpha_0)vT(t)}{t(1 - 2\alpha_1 vT(t))} dt = \int^{T(z)} \frac{(1 - (2\alpha_1 - \alpha_0)vT)(1 - 2\alpha_1 T)}{T(1 - 2\alpha_1 vT)(1 - (2\alpha_1 - \alpha_0)T)} dT \\
&= \int^{T(z)} \left(\frac{1}{T} + \frac{\alpha_0(v-1)}{\left(1 - \frac{2\alpha_1 v}{2\alpha_1 - \alpha_0}\right)(1 - (2\alpha_1 - \alpha_0)T)} \right. \\
&\qquad \left. + \frac{\alpha_0(v-1)}{\left(1 - \frac{2\alpha_1 - \alpha_0}{2\alpha_1 v}\right)(1 - 2\alpha_1 vT)} \right) dT \\
&= \log(T(z)) + \frac{\alpha_0(v-1)}{(2\alpha_1 - \alpha_0 - 2\alpha_1 v)} \log\left(\frac{1}{1 - (2\alpha_1 - \alpha_0)T(z)}\right) \\
&\qquad + \frac{\alpha_0(v-1)}{(2\alpha_1 v - (2\alpha_1 - \alpha_0))} \log\left(\frac{1}{1 - 2\alpha_1 vT(z)}\right) + \tilde{C}(v),
\end{aligned}$$

or

$$M(z,v) = C(v)T(z) \left(\frac{1 - (2\alpha_1 - \alpha_0)T(z)}{1 - 2\alpha_1 vT(z)}\right)^{\frac{\alpha_0(v-1)}{2\alpha_1 v - (2\alpha_1 - \alpha_0)}}.$$

Since $\frac{\partial}{\partial z} M(z,v)\big|_{z=0} = T_1 = C(v)T_1$, it follows further $C(v) = 1$, and we get thus as solution of the generating function $M(z,v)$:

$$M(z,v) = T(z) \left(\frac{1 - (2\alpha_1 - \alpha_0)T(z)}{1 - 2\alpha_1 vT(z)}\right)^{\frac{\alpha_0(v-1)}{2\alpha_1 v - (2\alpha_1 - \alpha_0)}}. \tag{18}$$

The asymptotic behaviour of every factorial moment $\mathbb{E}(X_n^{\underline{r}}) := \sum_{m \geq 0} m^{\underline{r}} \mathbb{P}\{X_n = m\}$, with $m^{\underline{r}} := m(m-1)\cdots(m-r+1)$, is studied by substituting $w := v - 1$ and extracting coefficients of w^r, where we use

$$[w^r]M(z, 1+w) = \sum_{n \geq 1} \sum_{m \geq r} T_n \frac{m^{\underline{r}}}{r!} \mathbb{P}\{X_n = m\} z^n. \tag{19}$$

We obtain by expanding (18)

$$M(z,1+w) = T(z)\exp\Big(w\Big[\sum_{k\geq 0}(-1)^k\left(\frac{2\alpha_1}{\alpha_0}\right)^k w^k\Big]\times\Big[\log\left(\frac{1-(2\alpha_1-\alpha_0)T(z)}{1-2\alpha_1T(z)}\right)+ +\sum_{k\geq 1}\left(\frac{2\alpha_1T(z)}{1-2\alpha_1T(z)}\right)^k\frac{w^k}{k}\Big]\Big),$$

and thus

$$[w^r]M(z,1+w) = T(z)\sum_{l=0}^{r}[w^{r-l}]\frac{1}{l!}\left(\sum_{k\geq 0}(-1)^k\left(\frac{2\alpha_1}{\alpha_0}\right)^k w^k\right)^l\times \times\left(\log\left(\frac{1-(2\alpha_1-\alpha_0)T(z)}{1-2\alpha_1T(z)}\right)+\sum_{k\geq 1}\left(\frac{2\alpha_1T(z)}{1-2\alpha_1T(z)}\right)^k\frac{w^k}{k}\right)^l. \tag{20}$$

In order to get the asymptotic behaviour of the coefficients $[z^nw^r]M(z,v)$, we will expand (20) around the dominant singularity $z=\rho=\frac{\left(\frac{\alpha_0}{2\alpha_1}\right)^{\frac{\alpha_0}{2\alpha_1-\alpha_0}}}{2\varphi_0\alpha_1}$, which is equivalent to expand around $T(z)=\tau=\frac{1}{2\alpha_1}$. We obtain

$$\begin{aligned}
[w^r]M(z,1+w) &= T(z)[w^{r-1}]\left(\sum_{k\geq 0}(-1)^k\left(\frac{2\alpha_1}{\alpha_0}\right)^k w^k\right)\cdot \\
&\cdot\left(\log\left(\frac{1-(2\alpha_1-\alpha_0)T(z)}{1-2\alpha_1T(z)}\right)+\sum_{k\geq 1}\left(\frac{2\alpha_1T(z)}{1-2\alpha_1T(z)}\right)^k\frac{w^k}{k}\right) \\
&+\frac{T(z)}{2}[w^{r-2}]\left(\sum_{k\geq 0}(-1)^k\left(\frac{2\alpha_1}{\alpha_0}\right)^k w^k\right)^2\cdot \\
&\cdot\left(\log\left(\frac{1-(2\alpha_1-\alpha_0)T(z)}{1-2\alpha_1T(z)}\right)+\sum_{k\geq 1}\left(\frac{2\alpha_1T(z)}{1-2\alpha_1T(z)}\right)^k\frac{w^k}{k}\right)^2 \\
&+\mathcal{O}\left(\log^2\left(\frac{1}{1-2\alpha_1T(z)}\right)\frac{1}{(1-2\alpha_1T(z))^{r-3}}\right),
\end{aligned}$$

which leads to

$$
\begin{aligned}
[w^r]M(z,1+w) &= \frac{T(z)}{r-1}\left(\frac{2\alpha_1 T(z)}{1-2\alpha_1 T(z)}\right)^{r-1} \\
&\quad + \mathcal{O}\left(\log\left(\frac{1}{1-2\alpha_1 T(z)}\right)\frac{1}{(1-2\alpha_1 T(z))^{r-2}}\right), \quad \text{for } r \geq 3, \\
[w^2]M(z,1+w) &= T(z)\left(\frac{2\alpha_1 T(z)}{1-2\alpha_1 T(z)}\right) \\
&\quad + \mathcal{O}\left(\log^2\left(\frac{1}{1-2\alpha_1 T(z)}\right)\right), \\
[w^1]M(z,1+w) &= T(z)\log\left(\frac{1-(2\alpha_1-\alpha_0)T(z)}{1-2\alpha_1 T(z)}\right).
\end{aligned}
$$

Singularity analysis of generating functions gives then the following asymptotic expansions.

$$
\begin{aligned}
[z^n w^r]M(z,1+w) &= \frac{\left(\frac{\alpha_1}{\alpha_0}\right)^{\frac{r-1}{2}}\left(\varphi_0\alpha_1\left(\frac{2\alpha_1}{\alpha_0}\right)^{\frac{\alpha_0}{2\alpha_1-\alpha_0}}\right)^n n^{\frac{r-3}{2}}}{2\alpha_1(r-1)\Gamma\left(\frac{r-1}{2}\right)}\left(1+\mathcal{O}\Big(\frac{\log n}{\sqrt{n}}\Big)\right), \\
&\quad \text{for } r \geq 3, \\
[z^n w^2]M(z,1+w) &= \frac{\sqrt{\frac{\alpha_1}{\alpha_0}}\left(\varphi_0\alpha_1\left(\frac{2\alpha_1}{\alpha_0}\right)^{\frac{\alpha_0}{2\alpha_1-\alpha_0}}\right)^n n^{-\frac{1}{2}}}{2\alpha_1\Gamma\left(\frac{1}{2}\right)}\left(1+\mathcal{O}\Big(\frac{\log n}{\sqrt{n}}\Big)\right), \\
[z^n w^1]M(z,1+w) &= \frac{\left(\varphi_0\alpha_1\left(\frac{2\alpha_1}{\alpha_0}\right)^{\frac{\alpha_0}{2\alpha_1-\alpha_0}}\right)^n}{4\alpha_1 n}\left(1+\mathcal{O}\Big(\frac{\log n}{\sqrt{n}}\Big)\right).
\end{aligned} \tag{21}
$$

By an application of the reflection law of the Gamma-function

$$\Gamma\left(\frac{r}{2}+1\right)\Gamma\left(\frac{r-1}{2}\right) = \frac{r!\sqrt{\pi}}{(r-1)2^{r-1}},$$

we obtain from (21) for every fixed $r \geq 1$ the expansion

$$
\begin{aligned}
&[z^n w^r]M(z,1+w) = \\
&\quad = \frac{\left(\frac{\alpha_1}{\alpha_0}\right)^{\frac{r-1}{2}} 2^{r-1}\Gamma\left(\frac{r}{2}+1\right)\left(\varphi_0\alpha_1\left(\frac{2\alpha_1}{\alpha_0}\right)^{\frac{\alpha_0}{2\alpha_1-\alpha_0}}\right)^n n^{\frac{r-3}{2}}}{2\alpha_1\sqrt{\pi}\, r!}\left(1+\mathcal{O}\Big(\frac{\log n}{\sqrt{n}}\Big)\right).
\end{aligned}
$$

Due to formula (3), we also have the asymptotic expansion

$$T_n = \frac{\sqrt{\alpha_0}}{4\alpha_1^{\frac{3}{2}}\sqrt{\pi}}\left(2\varphi_0\alpha_1\left(\frac{2\alpha_1}{\alpha_0}\right)^{\frac{\alpha_0}{2\alpha_1-\alpha_0}}\right)^n n^{-\frac{3}{2}}\left(1+\mathcal{O}\Big(\frac{1}{n}\Big)\right),$$

and by using (19), we obtain as asymptotic expansion for every factorial moment

$$\mathbb{E}(X_n^{\underline{r}}) = 2^{\frac{r}{2}} \left(\frac{2\alpha_1}{\alpha_0}\right)^{\frac{r}{2}} \Gamma\left(\frac{r}{2}+1\right) n^{\frac{r}{2}} \left(1 + \mathcal{O}\Big(\frac{\log n}{\sqrt{n}}\Big)\right). \tag{22}$$

With the estimate $\mathbb{E}(X_n^r) = \mathbb{E}(X_n^{\underline{r}}) + \mathcal{O}\big(\mathbb{E}(X_n^{\underline{r-1}})\big)$, we obtain that the asymptotic expansion (22) also holds for every (ordinary) r-th moment $\mathbb{E}(X_n^r)$. An application of the Theorem of Fréchet and Shohat shows finally, that $\frac{X_n}{\sqrt{n}}$ is asymptotically Rayleigh distributed with parameter $\beta = \sqrt{\frac{2\alpha_1}{\alpha_0}} = \sqrt{\frac{\tau\varphi''(\tau)}{\varphi'(\tau)}}$ and Theorem 2 is proven for CASE C.

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