COMPUTER-FREE EVALUATION OF A DOUBLE INFINITE SUM VIA EULER SUMS

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ABSTRACT. A short and computer-free proof (using Euler sums and multiple zeta functions) is provided for a double sum that was recently computed by Pemantle and Schneider using the computer software Sigma.

1. INTRODUCTION

The evaluation of the double sum

$$S := \sum_{j,k=1}^{\infty} \frac{H_j(H_{k+1}-1)}{jk(k+1)(j+k)} = -\zeta(2) - 2\zeta(3) + 4\zeta(2)\zeta(3) + 2\zeta(5), \tag{1}$$

(where $H_n := \sum_{k=1}^{n} \frac{1}{k}$ denote harmonic numbers) appears in [4] and was obtained using Carsten Schneider's software Sigma. Here, we will give a short proof that is completely computer-free. It uses Euler sums and multiple zeta functions.

The second order harmonic numbers which appear in the sequel are denoted by $H_n^{(2)} := \sum_{k=1}^{n} \frac{1}{k^2}$.

We split the sum S and apply partial fraction decomposition:

$$S = \sum_{k\ge1} \frac{H_{k+1}-1}{k(k+1)} \sum_{j\ge1} \frac{H_j}{j(j+k)} = \sum_{k\ge1} \frac{H_{k+1}-1}{k^2(k+1)} \sum_{j\ge1} H_j \Big(\frac{1}{j} - \frac{1}{j+k}\Big).$$

The inner sum is simplified as follows:

$$\sum_{j\ge1} H_j \Big(\frac{1}{j} - \frac{1}{j+k}\Big) = \sum_{j\ge1} \Big(\frac{1}{j} - \frac{1}{j+k}\Big) \sum_{l=1}^{j} \frac{1}{l} = \sum_{l\ge1} \frac{1}{l} \sum_{j\ge l} \Big(\frac{1}{j} - \frac{1}{j+k}\Big).$$

The second sum here telescopes, and only k summands remain:

$$\begin{aligned}
\sum_{l\ge1} \frac{1}{l} \sum_{j\ge l} \Big(\frac{1}{j} - \frac{1}{j+k}\Big) &= \sum_{l\ge1} \frac{1}{l} \sum_{j=0}^{k-1} \frac{1}{l+j} = \sum_{l\ge1} \frac{1}{l^2} + \sum_{j=1}^{k-1} \sum_{l\ge1} \frac{1}{l(l+j)} \\
&= \zeta(2) + \sum_{j=1}^{k-1} \frac{1}{j} \sum_{l\ge1} \Big(\frac{1}{l} - \frac{1}{l+j}\Big) \qquad \text{[partial fraction decomposition]} \\
&= \zeta(2) + \sum_{j=1}^{k-1} \frac{H_j}{j} \qquad \text{[again by telescoping]} \\
&= \zeta(2) + \frac{H_k^2 + H_k^{(2)}}{2} - \frac{H_k}{k}.
\end{aligned}$$

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Thus the task reduces to evaluate the following single sum:

$$S=\sum_{k=1}^{\infty}\frac{H_{k+1}-1}{k^2(k+1)}\Big(\zeta(2)+\frac{H_k^2+H_k^{(2)}}{2}-\frac{H_k}{k}\Big). \tag{2}$$

After partial fraction expansion (and shifting the index if necessary), sum S can be written as follows:

$$S=-2\zeta(2)+\frac{1}{2}\sum_{k\ge1}\frac{H_kH_k^{(2)}}{k^2}+\frac{1}{2}\sum_{k\ge1}\frac{H_k^3}{k^2}-\sum_{k\ge1}\frac{H_k^2}{k^3}+(\zeta(2)-1)\sum_{k\ge1}\frac{H_k}{k^2}-2\sum_{k\ge1}\frac{1}{k^2}.$$

For the final computation of S we require the following evaluations of Euler sums via values of the Riemann zeta function:

$$\sum_{k\ge1}\frac{H_kH_k^{(2)}}{k^2}=\zeta(5)+\zeta(2)\zeta(3), \tag{3a}$$

$$\sum_{k\ge1}\frac{H_k^3}{k^2}=10\zeta(5)+\zeta(2)\zeta(3), \tag{3b}$$

$$\sum_{k\ge1}\frac{H_k^2}{k^3}=\frac{7}{2}\zeta(5)-\zeta(2)\zeta(3), \tag{3c}$$

$$\sum_{k\ge1}\frac{H_k}{k^2}=2\zeta(3), \tag{3d}$$

from which equation (1) then follows.

Equations (3c) and (3d) can be found explicitly in [3]. In [3] one also finds the identities

$$\sum_{k\ge1}\frac{H_k^3}{(k+1)^2}=\frac{15}{2}\zeta(5)+\zeta(2)\zeta(3), \tag{4a}$$

$$\sum_{k\ge1}\frac{H_k}{k^4}=3\zeta(5)-\zeta(2)\zeta(3), \tag{4b}$$

and due to

$$\sum_{k\ge1}\frac{H_k^3}{k^2}=\sum_{k\ge1}\frac{H_k^3}{(k+1)^2}+3\sum_{k\ge1}\frac{H_k^2}{k^3}-3\sum_{k\ge1}\frac{H_k}{k^4}+\sum_{k\ge1}\frac{1}{k^5},$$

we obtain equation (3b) as well, using (4a), (4b) and (3c).

To show (3a) we will apply Theorem 2 of [2], which gives

$$\sum_{k\ge1}\frac{H_{k-1}H_{k-1}^{(2)}}{k^2}=\zeta(2,1,2)+\zeta(2,2,1)+\zeta(2,3),$$

where the *multiple zeta functions* are defined by

$$\zeta(a_1,a_2,\dots,a_m)=\sum_{k_1>k_2>\cdots>k_m\ge1}\frac{1}{k_1^{a_1}k_2^{a_2}\cdots k_m^{a_m}}.$$

Since

$$\begin{aligned}\sum_{k\ge1}\frac{H_kH_k^{(2)}}{k^2} &= \sum_{k\ge1}\frac{H_{k-1}H_{k-1}^{(2)}}{k^2}+\sum_{k\ge1}\frac{H_{k-1}^{(2)}}{k^3}+\sum_{k\ge1}\frac{H_{k-1}}{k^4}+\sum_{k\ge1}\frac{1}{k^5}\\ &= \sum_{k\ge1}\frac{H_{k-1}H_{k-1}^{(2)}}{k^2}+\sum_{k>l\ge1}\frac{1}{k^3l^2}+\sum_{k>l\ge1}\frac{1}{k^4l}+\sum_{k\ge1}\frac{1}{k^5},\end{aligned}$$

we obtain

$$\sum_{k\ge1}\frac{H_kH_k^{(2)}}{k^2}=\zeta(2,1,2)+\zeta(2,2,1)+\zeta(2,3)+\zeta(3,2)+\zeta(4,1)+\zeta(5).$$

Using the following evaluations of the multiple zeta function given in [1] resp. [2]:

$$\begin{aligned}\zeta(2,1,2)=\zeta(2,3)&=\frac{9}{2}\zeta(5)-2\zeta(2)\zeta(3),\\ \zeta(2,2,1)=\zeta(3,2)&=-\frac{11}{2}+3\zeta(2)\zeta(3),\\ \zeta(4,1)&=2\zeta(5)-\zeta(2)\zeta(3),\end{aligned}$$

equation (3a) follows.

References

[1] D. Borwein, J. Borwein and R. Girgensohn, Explicit evaluation of Euler sums, *Proceedings of the Edinburgh Mathematical Society* 38, 277–294, 1995.

[2] J. Borwein and R. Girgensohn, Evaluation of triple Euler sums, *Electronic Journal of Combinatorics* 3, research paper #23, 1996.

[3] P. Flajolet and B. Salvy, Euler sums and contour integral representations, *Experimental Mathematics* 7, 15–35, 1998.

[4] R. Pemantle and C. Schneider, When is 0.999... equal to 1?, preprint, 2004.

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