Chapter 9

Vector rogue waves driven by polarisation instabilities

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9.1 Introduction

Over the last two decades, waves with anomalous amplitude (rogue waves) have been a focus of researchers from different fields ranging from oceanography to laser physics [1–23]. In the scientific and social context, revealing underlying, hidden mechanisms of rogue wave (RW) emergence may enable forecasting and localisation of the source of the extreme events and so reduce their detrimental impact on society. To quantify RWs, the following criteria have been suggested: (i) probability distribution functions for RW amplitudes should have ‘fat’ tails significantly deviating from Gaussian or Rayleigh distributions; and (ii) RWs should have amplitudes more than twice as large as the mean amplitude of the highest third of the waves (significant wave height), or RW amplitude should exceed the standard deviation of the ocean’s surface variations by more than eight times [3].

The major obstacle to studying rogue waves in the context of prediction and mitigation are the scarcity of these rare events and the inability to perform full-scale experiments on RW emergence in real-world scenarios. Mode-locked fibre lasers, as a source of pico- and femtosecond pulses with MHz repetition rates, provide a unique opportunity to observe more data on rogue waves in the short term (compared to the timescale of RWs in other systems such as in the ocean and financial markets [1–3]) and under laboratory-controlled conditions [7–12, 15–23]. Previously it has been found that RWs can be generated in mode-locked lasers in the timescale of a roundtrip [8–12] due to soliton–soliton interaction through the overlapping of their tails or soliton–dispersive wave interaction [24, 25]. The result of these interactions is coupling enhancement that leads to chaotic pulse bunching (soliton rain) in the timescale of a roundtrip [7–12]. This makes the scenario of the RW emergence similar to the dynamics of the coupled nonlinear oscillators, where

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desynchronisation causes random relative phase slips leading to the appearance of rogue waves [26–28].

Models of networks of coupled oscillators are widely used to study different synchronisation scenarios, from phase locking to chaotic phase drift, in the context of applications from biomedicine to laser physics [26–28]. For the phase-coupled oscillators, evolution can be considered based on the Kuramoto equation:

$$\frac{d\phi_i}{dt} = \Omega_i + \frac{K}{N} \sum_{j=1, j \neq i}^{N} G(\phi_i - \phi_j).$$  \hspace{1cm} (9.1)

Here, $\phi_i$ and $\Omega_i$ are the phase and the frequency of oscillator, $K$ is the coupling strength, and $G$ is the coupling function [26–28]. In the simplest case of two coupled oscillators with $G = \sin(\phi_i - \phi_j)$, equation (9.1) is called the Adler equation [26–28]. For a comprehensive study of the synchronisation scenarios in lasers, the amplitude dynamics along with dynamics of the population inversion have to be included in consideration, which results in a new class of coupled system, i.e., coupled oscillators with dynamic coupling strength [27]. With accounting for the vector nature of pump and lasing field interaction with active medium, orthogonal states of polarisation can be treated as coupled oscillators as well [29–35]. This provides a very good test bed for the study of synchronisation and desynchronisation of the longitudinal modes and the orthogonal states of polarisation (SOPs) in a fibre laser as a function of the laser parameters (power and ellipticity of the pump wave, and in-cavity birefringence and twist) tunable by the pump laser current driver, polarisation controller for the pump wave, and in-cavity polarisation controller [29–35]. Here, for erbium-doped lasers mode-locked with carbon nanotubes, we review our recent experimental and theoretical results and demonstrate that coupling of orthogonal SOPs provides an additional soliton–soliton and soliton–dispersive wave interaction mechanism leading to the RW emergence [16, 17]. To isolate such interaction from the contribution of the soliton–soliton and soliton–dissipative wave interactions, we used a low-pass filter with 1 MHz bandwidth. As a result, we found a new type of RW emerging at a slow timescale (tens of thousands of roundtrips) in the form of slow bright and dark rogue waves (BDRWs). With the help of a new vector model, we obtained theoretical results which correspond to the experimental data. Our theoretical analysis demonstrates that orthogonally polarised SOPs can be considered as quasi-equilibrium points, where a typical trajectory traverses the neighbourhood of one of the orthogonal SOPs with further switching to the other SOP. The dwelling time for the trajectory near each orthogonal SOP is defined by the cavity anisotropy, i.e., by the pump wave’s ellipticity adjustable with the help of the pump polarisation controller [17]. The escape from the neighbourhood of each SOP is driven by the in-cavity birefringence, tunable with the help of the in-cavity polarisation controller, and leads to the output power distribution satisfying the BDRW criteria. In addition to the aforementioned results, we demonstrate that, for a long fibre laser without generic mode-locking mechanism, the tuning of in-cavity fibre birefringence and twist can result in different synchronisation scenarios ranging from mode locking to random multi-pulsing satisfying rogue wave criteria [18–22].
The chapter is organised as follows. Section 9.2 is dedicated to the experimental and theoretical study of rogue waves driven by polarisation instability in Er-doped fibre lasers mode-locked with carbon nanotubes. In Section 9.3, for a long-cavity (615 m) Er-doped fibre laser, we provide analysis of the longitudinal mode synchronisation and desynchronisation caused by tuning of in-cavity fibre birefringence and twist.

9.2 Bright and dark rogue waves in mode-locked fibre laser

9.2.1 Experimental results

The laser setup is shown in Figure 9.1. Details of the setup specification are given in annex I. The fast dynamics comprises 11 oscillograms with 20 K samples in each, and a 25 ps sample interval recorded(40,245),(448,275) with help of a fast photodetector and oscilloscope. Slow dynamics includes 20 traces with 1024 samples in each and 1 μs sample interval obtained with the help of a polarimeter. The polarisation controllers (POC1 and POC2) have been tuned to find conditions for RW emergence. The additional results obtained, with the help of autocorrelator and optical spectrum analyser, are discussed in annex I.

The results for the fast dynamics are shown in Figures 9.2(a1) and (b1) for the pump currents \( I_p = 140 \text{ mA} \) and Figures 9.2(a2) and (b2) for 170 mA [16]. As shown in Figures 9.2(a1) and (a2), the voltage offset in the oscilloscope results in minimum voltage taking the negative values. The multi-pulse dynamics in Figures 9.2(a1) and (a2) are similar to the soliton rain, with the main difference in low pump power of 67 mW \( (I_p = 140 \text{ mA}) \) used here as compared to the 800 mW used by the other authors [8–12]. When pump current is increased to 170 mA, the number of pulses in a bunch increases (Figure 9.2(a2)). For both cases, the dynamics are stable within each frame, changing from frame to frame. Probability distribution histograms are shown in Figures 9.2(b1) and (b2). Accounting for output amplitude normalisation, RW criterion shows as \( V_n > 8 \), and so both cases shown in Figures 9.2(b1) and (b2) satisfy such criteria. Soliton–soliton interaction leading to RW appearance in the form of soliton rain on the scale of a few roundtrips is discussed in annex I [16]. As the total number of roundtrips measured by oscilloscope is 171, a low-pass filter

Figure 9.1. Schematic of a fibre laser mode-locked with carbon nanotubes: erbium-doped fibre (EDF); optical isolator (OISO); laser diode (LD); 980/1550 wavelength division multiplexer (WDM); carbon-nanotube-based mode locker (CNT); polarisation controllers (POC1 and POC2); output coupler (output C). Reproduced from [17].

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cannot be applied to filter out such interactions to reveal a slow dynamics at the timescale of 30 roundtrips.

To reveal the RW emergence at the slow timescales of 1 μs–20 ms (660 K roundtrips), we used a polarimeter to record 20 traces with 1024 samples in each and 1 μs sample interval. As a result, we found the degree of polarisation (DOP), output power, and normalised Stokes parameters \( s_1, s_2, s_3 \), which are related to the output powers of two linearly cross-polarised SOPs \( I_x \) and \( I_y \), and the phase difference between them \( \Delta \phi \) as follows:

\[
S_0 = I_x + I_y, \quad S_1 = I_x - I_y, \quad S_2 = 2\sqrt{I_x I_y} \cos \varphi, \quad S_3 = 2\sqrt{I_x I_y} \sin \Delta \phi, \\
\quad s_i = \frac{S_i}{\sqrt{S_1^2 + S_2^2 + S_3^2}}, \quad DOP = \frac{\sqrt{S_1^2 + S_2^2 + S_3^2}}{S_0}, \quad (i = 1, 2, 3).
\]

The obtained results are shown in figure 9.3. As follows from figure 9.3(a1) for \( J_p = 140 \) mA, the spikes in the output power correspond to the spikes in DOP and the phase difference jumps in \( \pi \). This means that the spikes are result of the random switching between orthogonally polarised SOPs as shown in figures 9.3(b1) and (c1). By increasing pump current to \( J_p = 170 \) mA and tuning POC2, we obtained results whereby the output power oscillated without the presence of RW events (figure 9.3(a2)). Thus, a high degree of polarisation and oscillating (between 0 and \( \pi \)) rather than randomly evolving phase difference can be considered to be a signature of the absence of RW events.

Probability distribution histograms (PDH) for the total output power \( I_x + I_y \), output powers \( I_x \) and \( I_y \) for the pump currents of 140 and 170 mA, are shown in figures 9.3(d1)–(f1) and (d2)–(f2). As follows from figures 9.3(d1)–(f1), the PDHs demonstrate the presence of so-called ‘bright’ (spikes in figure 9.3(a1)) and ‘dark’ (dips in figure 9.3(a1)) [13, 14] RWs driven by the interaction of two orthogonal
SOPs. The PDHs in figures 9.3(e1) and (f1) demonstrate that there are no RW statistics for the output powers \( I_x \) and \( I_p \). The ‘bright’ and ‘dark’ RW emergence is a result of interaction of the orthogonal SOP leading to the ‘antiphase’ dynamics of the output powers \( I_x \) and \( I_p \) (figure 9.3(b1)). In addition, we found that there is no RW in the case of regular phase difference oscillations between 0 and \( \pi \) (figures 9.3(d2)–(f2)). Thus, the experimental data for the fast (roundtrip scale) and slow (tens of thousands of roundtrips) timescales demonstrate: (i) soliton rain (fast timescale) and ‘bright’ and ‘dark’ RWs (slow timescale) for \( I_p = 140 \) mA, and (ii) soliton rain (fast timescale) and no RWs (slow timescale) for \( I_p = 170 \) mA.

### 9.2.2 Theoretical results

To justify that the RW emergence at the slow timescale is driven by polarisation instabilities, we use the vector model of an erbium-doped fibre laser mode-locked with CNT developed by Sergeyev and co-workers [16, 17, 29, 30, 35]. The model describes the evolution of the lasing field for two orthogonal SOPs \( (u = \sqrt{I_x} \exp(i\phi_x), v = \sqrt{I_y} \exp(i\phi_y) \) averaged over the pulse width and the population of the first excited level in the Er\(^{3+}\)-doped active medium [16, 17, 29, 30, 35]. We
introduced a slow-time variable $t_s = z(t_s t_R)$, where $t_R = L/V_p$ is the photon roundtrip time and $L$ is the cavity length, and assumed an ansatz in the form $E_{x, y}(t, t_s) = u(t_s)\text{sech}(t/T_p)$, $E_{x, y}(t, t_s) = v(t_s)\text{sech}(t/T_p)$ (where $T_p$ is the pulse width). After averaging over the time $T_p \ll t \ll t_R$, we obtained the following equations [16, 17, 29, 30, 35]:

$$\frac{d|u|}{dt} = \left(\frac{\alpha_l f_1 + \beta}{1 + \Delta^2} - \alpha_l - F(|u|, |v|)\right)|u| + \frac{\alpha_f \Delta}{1 + \Delta^2} |v|\cos(\Delta \varphi)$$

$$- \frac{2\alpha_f \Delta}{1 + \Delta^2} |v|\sin(\Delta \varphi) + \frac{2\gamma}{3} |v|^3 |u|\sin(2\Delta \varphi),$$

$$\frac{d|v|}{dt} = \left(\frac{\alpha_l f_1 - \beta}{1 + \Delta^2} - \alpha_l - F(|u|, |v|)\right)|v| + \frac{\alpha_f \Delta}{1 + \Delta^2} |u|\cos(\Delta \varphi)$$

$$+ \frac{2\alpha_f \Delta}{1 + \Delta^2} |u|\sin(\Delta \varphi) - \frac{2\gamma}{3} |u|^3 |v|\sin(2\Delta \varphi),$$

$$\frac{d\Delta \varphi}{dt} = \Delta \Omega + K_{NL}\sin^2(\Delta \varphi) + K_{NL}\sin(\Delta \varphi) + K_{as}\cos(\Delta \varphi),$$

$$\frac{d\tilde{J}_p}{dt} = \left[\frac{\alpha_l f_1 - \beta}{2} - \left(1 + \frac{I_p}{2} + d_3 S_0\right) f_1 - \left(d_3 S_1 + \frac{I_p (1 - \delta^2)}{2 (1 + \delta^2)}\right) f_2 - d_3 S f_3\right].$$

$$\frac{d\tilde{J}_p}{dt} = \left[\frac{1 - \delta^2}{1 + \delta^2} \frac{I_p (\alpha_l f_1 - \beta)}{4} - \left(1 + \frac{I_p}{2} + d_3 S_0\right) f_2 - \left(1 - \delta^2\right) I_p \frac{f_1}{2} + d_3 S_0\right].$$

$$\frac{d\tilde{J}_p}{dt} = -\left[\frac{d_3 S f_3}{2} + \left(1 + \frac{I_p}{2} + d_3 S_0\right) f_1\right].$$

$$\Delta \Omega = \frac{2\alpha_f \Delta}{1 + \Delta^2} - 2\beta, K_{NL} = \frac{\gamma L_{ir}}{6} (|v|^2 - |u|^2),$$

$$K_s = \frac{2(|v|^2 + |u|^2)}{|u||v|} \frac{\alpha_f \Delta}{1 + \Delta^2}, K_{as} = \frac{2(|v|^2 - |u|^2)}{|u||v|} \frac{\alpha_l f_1 \Delta}{1 + \Delta^2}.$$

$$F(|u|, |v|) = \frac{\alpha_l}{\pi} \left(1 - \frac{\alpha_l (|u|^2 + |v|^2)}{1 + \alpha_l (|u|^2 + |v|^2)} \arctan \left(\frac{\alpha_l (|u|^2 + |v|^2)}{1 + \alpha_l (|u|^2 + |v|^2)}\right)\right).$$

Here, time is normalised to the roundtrip time; pump and lasing powers ($I_p$ and $|u|^2$, $|v|^2$) are normalised to the corresponding saturation powers [16, 17, 29, 30, 35]. $\alpha_l$ is the total absorption of erbium ions at the lasing wavelength; $\alpha_0$ is the CNT absorption coefficient at the lasing wavelength; $\alpha_s$ is the ratio of saturation powers for CNT and erbium-doped fibre; $\alpha_s$ is the total insertion losses in cavity; $\beta$ is the birefringence strength ($2\beta = 2\pi L_{Lb}$, where $L_{Lb}$ is the beat length); $\gamma = \gamma_k L_{ir}$ (where $\gamma_k$ is the Kerr coupling constant); $\delta$ is the ellipticity of the pump wave; $\varepsilon = \varepsilon_{Fw}/\varepsilon_{Fw}$ is the ratio of the roundtrip time $t_R$ to the lifetime of erbium ions at the first excited level $\tau_{Fw}$; $\chi = (\sigma_a (L) + \sigma_e (L))|\sigma_a (L)|$, where $\sigma_a (L)$ and $\sigma_e (L)$ are absorption and emission cross sections at the lasing wavelength; $\Delta$ is the detuning of the lasing wavelength with respect to the maximum of the gain spectrum (normalised to the gain spectral width); $d_1 = \chi((1 + \Delta^2));$ functions $f_i (i = 1, 2, 3)$ are related to the angular distribution.
of the dipole moments of the excited ions \( n(\theta) \) expanded into a Fourier series [16, 17, 29, 30, 35]:

\[
n(\theta) = \frac{n_0}{2} + \sum_{k=1}^{\infty} n_{1k} \cos(k\theta) + \sum_{k=1}^{\infty} n_{2k} \sin(k\theta), \quad f_1 = \left(\frac{n_{10}}{2} - 1\right) + \frac{n_{11}}{2}, \quad f_2 = \frac{n_{21}}{2}.
\]

Equation (9.4) uses the assumption that the dipole moments of the absorption and emission transitions for erbium-doped silica are located in the plane orthogonal to the direction of the light propagation. This approximation results in the finite dimensional system presented by equation (9.3), where only the \( n_0 \), \( n_{12} \), and \( n_{21} \) components contribute to the dynamics. In contrast, the more general assumption of the 3D orientation distribution of the dipole moments’ orientation [31–34] results in an infinite-dimensional system with the more complex expansion of the angular distribution of the dipole moments into series of the Wigner’s \( D_{nm} \) or spherical \( Y_{nm} \) functions [34].

We solved equation (9.3) numerically for 660 K roundtrips (20 ms) using parameters quite close to the experimental: \( \alpha_1 = 21.53 \), \( \alpha_0 = 0.136 \), \( \alpha_r = 1.8 \times 10^{-5} \), \( \alpha_2 = 2.533 \), \( \chi = 2.3 \), \( \Delta = 0.1 \), \( I_p = 75 \), \( \gamma = 2 \times 10^{-6} \), and \( \varepsilon = 10^{-4} \). The results for \( \delta = 0.85 \) (elliptically polarised pump), \( \beta = \pi \cdot 10^{-2} \), \( I_p = 75 \), and \( \delta = 1 \) (circularly polarised pump), \( \beta = 0.8\pi \cdot 10^{-2} \), \( I_p = 80 \) shown in figure 9.4 demonstrate the interplay between in-cavity birefringence and the ellipticity of the pump wave in the context of the RW emergence [15]. As follows from figure 9.4(a1), the dynamics for the elliptically polarised pump take the form of the chaotic oscillations with the probability distribution histogram satisfying rogue wave criteria. Low-pass filtering of the theoretical data over 30 roundtrips corresponds to the experimental characterisation based on a polarimeter with 1 MHz resolution (figure 9.5).

Figure 9.4. (a1)–(a2) Dynamics of the output power averaged over the pulse width; (b1)–(b2) probability distribution histograms (the output power I is normalised as shown in figure 9.2). (IN1–IN2) enlarged fragment of the output power dynamics at 100 and 150 µs intervals. Parameters: \( L = 75 \), \( \alpha_1 = 21.53 \), \( \alpha_0 = 0.136 \), \( \alpha_r = 1.8 \times 10^{-5} \), \( \alpha_2 = 2.533 \), \( \chi = 2.3 \), \( \Delta = 0.1 \), \( \gamma = 2 \times 10^{-6} \), \( \varepsilon = 10^{-4} \); (a1), (b1) \( \delta = 0.85 \) (elliptically polarised pump), \( \beta = \pi \cdot 10^{-2} \); (a2), (b2) \( \delta = 1 \) (circularly polarised pump), \( \beta = 0.8\pi \cdot 10^{-2} \).
The obtained results shown in figure 9.5 are in good correspondence with the experimental data (figure 9.3): the extreme spikes in the output power (figure 9.5(a1)) correspond to the spikes in DOP and phase difference jumps in \( \pi \) (figure 9.5(b1)) leading to the transitions between orthogonally polarised SOPs (figure 9.5(c1)). Asymmetrical probability distribution histograms for the powers \( I_x \) and \( I_y \) (figures 9.5(e1) and (f1)) and histogram for the total power \( I = I_x + I_y \) (figure 9.4(d1)) correspond to the histograms shown for \( I_p = 140 \) mA in polarimetry experiments (figure 9.3) in the context of presence of ‘dark’ and ‘bright’ RWs. For the circularly polarised pump (\( \delta = 1 \)) and \( I_p = 80 \), the dynamics takes the form of more regular oscillations as shown in figure 9.4(a2). Unlike the experimental data for \( I_p = 170 \) mA, the RWs are absent (figure 9.4(b2)) because the approximation of the average over the pulse output power neglects the contribution of the soliton rain in

Figure 9.5. (a)−(c) Dynamics after low-pass filtering based on a Hanning window with transmission spectrum \( T(f) = \left( 1 + \cos(\pi f / f_c) \right) / 2 \), \( f \ll f_c = 1 \) MHz. (a1)−(a2) dynamics of the output powers \( I_x \) (red), \( I_y \) (blue), \( I = I_x + I_y \) (black); (b1)−(b2) dynamics of the phase difference \( \Delta \Phi \) (red) and DOP (black); (c1)−(c2) trajectories on the Poincaré sphere; (d)−(i) probability distribution histograms for the total output power \( I = I_x + I_y \) (d1), \( I_x \) (d2), and \( I_y \) (d3). Each output power \( I_x (I, I_y, I) \) is normalised as shown in figure 9.2. Parameters: \( I_p = 75 \), \( \alpha_1 = 21.53 \), \( \alpha_2 = 0.136 \), \( \alpha_3 = 1.8 \times 10^{-2} \), \( \alpha_4 = 2.533 \), \( \chi = 2.3 \), \( \Delta = 0.1 \), \( \gamma = 2 \times 10^{-3} \), \( \sigma = 10^{-3} \), (a1)−(f1) \( \delta = 0.85 \) (elliptically polarised pump), \( \beta = 10^{-2} \); (a2)−(c2) \( \delta = 1 \) (circularly polarised pump), \( \beta = 0.8 \times 10^{-2} \).
the statistics of the output power. The results for low-pass filtering are shown in figure 9.4(a2). The antiphase dynamics of the output powers \( I_1 \) and \( I_2 \) results in the regular oscillations of the total output power, DOP, and the phase difference that is quite similar to the dynamics observed experimentally (figure 9.3). The difference in the theoretical and experimental data for output powers \( I_1 \) and \( I_2 \) at \( J_p = 140 \) mA can be explained by the different settings of the in-cavity polarisation controller that transforms the output SOP by different ways for \( J_p = 140 \) and 170 mA. The detailed explanation for this is found in annex I.

In the context of the synchronisation scenarios for the coupled nonlinear oscillators, the equation for the phase difference (equation 9.1) is a further generalisation of the Adler equation [26–28] accounting for asymmetry in the coupling of polarisation modes (\( K_{\alpha\beta} \) coefficient) and polarisation mode coupling based on the Kerr effect (\( K_{NL} \)). In addition, \( I_1 \) and \( I_2 \) are also time-dependent variables and coupling coefficients \( K_\alpha, K_{\alpha\beta}, K_{NL} \) depend on \( I_1 \) and \( I_2 \); this means we have a new class of coupled oscillators with time-dependent coupling strength [27]. It has been shown recently for such systems that by tuning the coupling strength and the oscillators’ frequency difference, it is possible to tune the phase dynamics from the phase locking to the phase oscillations and chaotic phase slips including RWs [15]. In our case, the orthogonally polarised SOPs can be considered as coupled oscillators where the frequency difference depends on the birefringent strength and the coupling strength is a function of the ellipticity and the power of the pump wave [15]. Thus, adjustment of the pump wave and in-cavity polarisation controllers (POC1 and POC2 in figure 9.1) and pump power result in tuning the frequency difference and coupling strength. This provides the complex dynamics tunability from the phase-difference oscillations (figures 9.3(b2) and 9.5(b2)) to chaotic phase jumps (figures 9.3(b1) and 9.5(b1)), resulting in ‘bright’ and ‘dark’ RW emergence (figures 9.3(a1) and (d1) and 9.5(a1) and (d1)) at the slow timescale.

9.3 Synchronisation and desynchronisation phenomena in a long cavity Er-doped fibre laser

9.3.1 Risken–Nummedal–Graham–Haken instability

Modulation instabilities, discovered between 50 and 150 years ago, have created a framework for studying the complexity of different wave phenomena including turbulence and rogue waves [36–53]. For example, multimode Risken–Nummedal–Graham–Haken (RNGH) instability leads to the existence of a second lasing threshold, the exceeding of which results in excitation of a large number of longitudinal spatial modes and generation of a pulse train with a period equal to the photon roundtrip time [45–53]. The history of RNGH instability started in 1968, and since then the extensive theoretical and experimental study of Er-doped fibre lasers has revealed a decreased second lasing threshold to values slightly exceeding the value for the first threshold [48–53]. In this section, we review our recent experimental and theoretical results and demonstrate a new type of multimode instability, namely vector resonance multimode instability (VRMI), that is behind
complex behaviours ranging from the turbulent, including different types of rogue waves, to stable pulse trains similar to the laser mode-locking regime [22, 23]. We have justified experimentally and theoretically that increased in-cavity birefringence strength causes spatial modulation of the SOP of the in-cavity lasing field (with a period related to the beat length) and thus leads through dispersion relation to the emergence of additional satellite frequencies with frequency splitting proportional to the birefringence strength. When the splitting approaches a fundamental frequency inversely proportional to the photon roundtrip time, parametric resonance results in longitudinal mode synchronisation and emergence of a pulse train with narrow pulse width. In contrast, in the case of weak birefringence and strong fibre twist adjusted with the help of an in-cavity polarisation controller, we demonstrate rogue wave emergence through transition from quasiperiodic motion (torus in the phase space) and further to chaotic behaviour satisfying RW criteria through the Ruelle–Takens–Newhouse scenario (route to chaos via two-dimensional torus destruction [54]).

9.3.2 Experimental data

The details of the schematic of the unidirectional cavity fibre laser are found in annex II [22]. The pump power for the first lasing threshold of the continuous wave operation was 16 mW, whereas the second threshold of the multimode instability was 18 mW. The paddles of the polarisation controllers (POC1 and POC2) were set with respect to the vertical position. The paddle of POC1 was oriented at $\theta_1 = -59^\circ$ and the paddle of POC2 was adjusted at three different positions ($\theta_2 = -78^\circ$, $-74^\circ$, $-69^\circ$). The radiofrequency (RF) spectrum comprises three peaks which were resolved better near the 1000th harmonic of the fundamental self-mode locking frequency, i.e. for 325.2 MHz (figure 9.6(a)): the central peak (1000th harmonic), two adjustable with the help of the POC2 satellites, and two satellites independent of POC2 orientation. As follows from the figures 9.6(a) and (b), the adjustment of the POC results in

![Figure 9.6.](image)

Figure 9.6. (a) RF spectrum, (b) oscillograms, and (c) probability distribution histograms (the output amplitude is normalised as shown in figure 9.2) for different setting of the POC2: $\theta_2 = -78^\circ$, $-74^\circ$, $-69^\circ$.  

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approaching of the satellites from the neighbouring to 325.2 MHz lines to the main peak of 325.2 MHz. Such adjustment results in regime stabilisation similar to mode-locking for the case when satellites match the main line (figures 9.6(a) and (b)). Unlike multimode RNGH instability [48–53] demonstrating oscillations close to the harmonic with the period of photon roundtrip time, the pulse train in figure 9.6 has a pulse width of 40 ns. This period is much less than the roundtrip time of 3 μs that indicates the excitation of many phase-synchronised longitudinal modes. Analysis of the amplitude distribution shown in figure 9.6(c) reveals that the complex oscillatory dynamics shown in figure 9.6(b) do not satisfy RW criteria.

The results on polarisation dynamics are shown in figure 9.7. As follows from figure 9.7(a), the size of the trajectory on the Poincaré sphere indicates that N-fold beat length deviates from the cavity length. Thus, N-fold beat length matches the cavity length when the trajectory is shrinking to the dot. This mismatch is accompanied by oscillations of the output power \( S_0 \) and phase difference \( \Delta \varphi \) (figures 9.7(a1) and (b1)). DOP of 60% indicates that the laser dynamics oscillate with a timescale smaller than 1 μs for the POC2 setting at \( \theta_2 = -78^\circ \) (figures 9.7(a1) and (b1)). Setting the POC2 at \( \theta_2 = -74^\circ \) demonstrates suppression of the fast dynamics. The constant outputs and high DOP (over 80%) in figures 9.7(a), (a3), and (b3) indicate stable mode locking (figure 9.6(b)) along with stable phase difference and SOP locking (figure 9.7(b3)). As follows from figure 9.7, all the aforementioned vector dynamics are localised nearby one SOP, and this may be a reason for the RW absence.

As follows from section 9.2, the random switching between orthogonal SOPs results in emergence of spikes with statistics satisfying RW criteria. This is why we readjust the POC1 and POC2 settings to enable switching between the orthogonal SOPs leading to RW emergence.

As follows from the obtained results (figure 9.8(a)), adjustment of POC1 and POC2 results in excitation complex multi-pulsing (middle figure) including soliton rain (bottom figure). The corresponding RF spectrum indicates excitation of the satellites with frequencies: \( f_1 = 4.8 \times 10^{-5} f_0 \), \( f_2 = f_1/4 \), \( f_3 = f_2/4 \) (figure 9.8(b)). As follows from the figures 9.8(a) and (c), though dynamics for all three cases

![Figure 9.7](image)

**Figure 9.7.** (a) trajectories in normalised Poincaré sphere: (a1)–(a3) output powers versus time: \( I_0 \) (red), \( I_2 \) (blue), and \( I = I_0 + I_2 \) (black); (b1)–(b3) phase differences \( \Delta \varphi/\pi \) (red) and DOP (black) versus time; POC2 settings: \( \theta_2 = -78^\circ, -74^\circ, -60^\circ \).
Figure 9.8. (a) Oscillograms, (b) RF spectrum, (c)–(e) probability distribution histograms (the output amplitude is normalised as shown in figure 9.2) for different setting of POC1 and POC2.

demonstrate the L-shape statistics typical for RWs, the case of mode-locking leads to predictable dynamics and so pulses cannot be counted as RWs.

The results for polarisation dynamics shown in figure 9.9 confirm the connection of the RW emergence with random switching between orthogonal SOPs. In addition, as follows from figures 9.7 and 9.9, complex dynamics including RWs exist if the output powers for the orthogonal SOPs are approximately equal, whereas the mode locking appears when the output power for one of the SOPs is close to zero, i.e. for linearly polarised SOPs. This means that, by controlling POC1 and POC2, it is possible to adjust anisotropy and in-cavity birefringence to enable strong orthogonal SOP coupling that, according to the theory of the coupled oscillators, leads to the synchronisation that in our case corresponds to the longitudinal mode locking. In contrast to the mode locking, weak coupling between the SOPs leads to phase-difference random slips and so to the emergence of spikes in the dynamics of the output power.

9.3.3 Theoretical results

The experimental results can be well understood based on the vector model of an Er-doped fibre laser. The following rate equations are derived from the model suggested by Sergeyev et al [22]:

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Figure 9.9. (a) Trajectories in normalised Poincaré sphere: (a1)–(a3) output powers versus time: $I_x$ (red), $I_y$ (blue), and $I = I_x + I_y$ (black); (b1)–(b3) phase differences $\Delta \phi / \pi$ (red) and DOP (black) versus time.

\[
\begin{align*}
\frac{\partial S_0}{\partial z} + \frac{\partial S_0}{\partial t} &= \left( \frac{2 \alpha f_1}{1 + \Delta^2} - 2 \alpha_2 \right) S_0 + \frac{2 \alpha f_2}{1 + \Delta^2} S_1 + \frac{2 \alpha_1 f_1}{1 + \Delta^2} S_2, \\
\frac{\partial S_1}{\partial z} + \frac{\partial S_1}{\partial t} &= \gamma S_2 S_3 + \left( \frac{2 \alpha f_1}{1 + \Delta^2} - 2 \alpha_2 \right) S_1 + \frac{2 \alpha f_2}{1 + \Delta^2} S_0 - \frac{2 \alpha_1 f_2}{1 + \Delta^2} S_3 + \rho S_2, \\
\frac{\partial S_2}{\partial z} + \frac{\partial S_2}{\partial t} &= -\gamma S_1 S_3 + \frac{2 \alpha f_1}{1 + \Delta^2} S_0 + \left( \frac{2 \alpha f_1}{1 + \Delta^2} - 2 \alpha_2 \right) S_2 + \left( \frac{2 \alpha f_2}{1 + \Delta^2} - 2 \rho \right) S_3 - \rho S_1, \\
\frac{\partial S_3}{\partial z} + \frac{\partial S_3}{\partial t} &= \frac{2 \alpha_1 f_1}{1 + \Delta^2} S_1 - \frac{2 \alpha_1 f_2}{1 + \Delta^2} S_2 + 2 \beta S_2 + \left( \frac{2 \alpha f_1}{1 + \Delta^2} - 2 \alpha_2 \right) S_3, \\
\frac{df_1}{dt} &= e \left( \frac{\Delta \phi_0}{2} - 1 - \left( 1 + \frac{I \rho f_2}{2} + d_0 S_0 \right) f_1 \right) - \left( d_0 S_1 + \frac{I \rho f_2}{2} \right) f_2 - d_0 S_3 f_3, \quad (9.5) \\
\frac{df_2}{dt} &= e \left[ \frac{I \rho f_2}{2} \left( \frac{\Delta \phi_0}{2} - 1 \right) - \left( \frac{I \rho f_2}{2} + 1 + d_0 S_0 \right) f_2 - \left( \frac{I \rho f_2}{2} \frac{f_1}{2} + d_0 S_1 \right) \right], \\
\frac{df_3}{dt} &= -e \left[ \frac{d_0 S_1 f_1}{2} + \left( \frac{I \rho f_2}{2} + 1 + d_0 S_0 \right) f_3 \right].
\end{align*}
\]
Equation (9.5) is the further generalisation of equation (9.3) in the context of accounting for the angle of the SOP rotation around the S_2-axis induced by POC2 adjustment (ρ). Pump at 1480 nm (8L_p = 2πL/L_0) with S-rop emitted by POC2 controller (2β = 2πL/L_0) and birefringence induced by polarisation hole burning in the active fibre Δ_p = 2α_1f_2Δ/1 + ∆^2). In the experiments, we set the pump power at 18 mW and so exclude mode locking based on nonlinear polarisation rotation that requires much larger power (approximately 800 mW) [8–12]. The pulse width in our experiments (figures 9.6 and 9.8) was estimated to be of 40 ns, and so the second-order dispersion can be neglected in the context of the qualitative description of the experiments. The dynamics of the medium polarisation were also ignored because the pulse width is also much longer than the time of transverse relaxation of 160 fs.

To specify conditions for the vector resonance mode locking, we linearise equation (9.5) in the vicinity of the steady-state solution \( F_0 = (S_{00}, ±S_{00}, 0, 0, f_{10}, f_{20}, 0)^T \) and find eigenvalues as follows (details are found in appendix II):

\[
\begin{align*}
(\text{I}) & \quad \lambda_0 = iq + A_0(q, I_p, \xi), \\
(\text{II}) & \quad \lambda_1 = A_1(q, I_p, \xi) + i\Omega_1(q, I_p, \xi), \\
 & \quad \lambda_2 = A_2(q, I_p, \xi) + i\Omega_2(q, I_p, \xi), \\
 & \quad \lambda_3 = A_3(q, I_p, \xi) + i\Omega_3(q, I_p, \xi), \\
(\text{III}) & \quad \lambda_4 = A_4(q, I_p, \beta, \xi) + i\Omega_4(q, I_p, \beta, \xi), \\
 & \quad \lambda_5 = A_5(q, I_p, \beta, \xi) + i(q + \Delta\Omega(q, I_p, \beta, \xi)), \\
 & \quad \lambda_6 = A_6(q, I_p, \beta, \xi) + i(q - \Delta\Omega(q, I_p, \beta, \xi)).
\end{align*}
\]

Here, \( q = 0, ±1, ±2, \ldots ±N \) is the wave number of the longitudinal mode. All eigenvalues are normalised to the fundamental frequency \( \omega = 2\pi f_R \). Here, we neglected the fibre twist, i.e. we use condition \( p = 0 \).

The obtained results are shown in figures 9.10(a)–(c) [22]. As follows from figure 9.10(a), the threshold of the multimode instability is the same for all longitudinal modes and coincides with the first lasing threshold for the circularly polarised pump and slightly exceeds the first one with increased pump anisotropy parameter \( \xi \). In contrast, threshold pump values for RGHM are different for excitation of different longitudinal modes. The threshold for multimode instability (branch I) is the same as the threshold of birefringence-dependent RF satellite excitation (branch III). As follows from figure 9.10(b), the scalar branch II comprises additional birefringence-independent satellites.
Figure 9.10. (a)–(c) Results of linear stability analysis of equation (9.5) in vicinity of steady solutions: (a1)–(c1) \( F_0 = (S_{00} \pm S_{00} f_{10} f_{20} 0 0 0) \) and (a2–b2) \( F_0 = (S_{00} 0 0 \pm S_{00} f_{10} 0 0 0) \). (a1) Vector multimode instability in terms of positive real parts of eigenvalues, i.e. \( \lambda_0 \) (dots), \( \lambda_1 \) (solid line) (\( \lambda_0 \) is close to the \( \lambda_1 \) and so is not shown here), and the output signal \( S_0 \) (dashed line) versus pump power \( I_p \) for \( \zeta = 0 \) (blue lines), \( \zeta = 0.01 \) (black line); (b1) frequencies of the scalar branch: \( \Omega_1 \) (empty circles), \( \Omega_2 \) (empty triangles), \( \Omega_3 \) (empty squares) and the vector branch \( \Omega_4 \) (empty diamonds) along with the real parts of the scalar branch: \( \Omega_1 \) (filled circles), \( \Omega_2 \) (filled triangles), \( \Omega_3 \) (filled squares) and the vector branch: \( \Omega_4 \) (filled diamonds) versus pump power \( I_p \) for \( \zeta = 0.1 \). (c1) Frequencies for the branch I, i.e. \( \Omega_4 = q \), \( q = 0.1, 0.2, 0.3 \) (solid lines), and the vector branch (III), i.e. \( \Omega_4 = q \pm \Delta \Omega \) (squares, triangles, circles) versus the birefringence strength \( 2 \beta = 2 \Delta n_f / f_{20} / (1 + \Delta \Delta) \) for \( \zeta = 0.1 \). The equality \( \Omega_k = \Omega_k + \Delta \) for \( 2 \beta = 2 \Delta n_f / f_{20} / (1 + \Delta \Delta) = 1.2 \). Results in the resonance mode locking shown in figure 9.6. All frequencies are normalised to the fundamental frequency. Parameters: \( L = 615 \text{ m}, a_1 = \ln(10) 6.4, a_2 = \ln(10) 0.5 \times 3 = 0.07, A = 0.1, I_p = 10 \text{ (c),} \gamma = 2 \times 10^{-4}, \beta = 1 \) (a, b), \( q = 1 \) (b, c), \( \zeta = 0.1 \) (b, c). Reprinted figure with permission from [22]. Copyright 2017.

which are located in the RF spectrum at 0.01\( f \) (\( f \) is the fundamental frequency) with respect to the longitudinal mode frequency \( q \). The satellites can be excited through parametric phase-locking with the frequencies of branch I. If we consider branch III, we can find that increased birefringence strength can result in resonance conditions where satellites frequencies for the longitudinal mode \( q \) (branch III) match the frequency of the longitudinal mode \( q + N \) (where \( N \) is integer) from branch I.

Thus, if we compare figures 9.6 and 9.10(a)–(c), we can conclude that the theoretical results are in a good correspondence with the experimental data, i.e. (i) the threshold of the multimode instability (the second threshold) slightly exceeds the first lasing threshold; (ii) if the birefringence-dependent satellites frequencies are off-resonance with the longitudinal mode frequencies, then the dynamics take the form of complex oscillations; (iii) resonance of the longitudinal modes’ frequencies with the birefringence-dependent satellites results in their synchronisation and so to the stable mode locking is accompanied by stable SOP locking.

As follows from the experimental data shown in figures 9.8 and 9.9, the dynamics nearby the orthogonal SOPs (with \( I_x = I_y \)), including random switching between them, results in complex multi-pulse dynamics RW emergence. To prove this, we linearise equation (9.5) in the vicinity of the steady-state solution \( F_0 = (S_{00} 0 0 \pm S_{00} f_{10} 0 0 0) \) and find eigenvalues as follows [23]:

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(I) \( \lambda_0 = i2\pi q \),

(II) \( \lambda^2 + (c_2 + i2\pi q)\lambda + b_1c_1 + i2\pi c_2q = 0 \),

(III) \( \lambda^2 + (c_2 + i2\pi q - i\rho)\lambda + \frac{b_1c_1}{2}(1 - i\alpha\Delta) + ic_2(2\pi q - \rho) = 0 \),

(IV) \( \lambda^2 + (c_2 + i2\pi q + i\rho)\lambda + \frac{b_1c_1}{2}(1 + i\alpha\Delta) + ic_2(2\pi q + \rho) = 0 \),

\[
b_1 = 2\alpha_1S_{00}, \quad c_1 = \varepsilon d_1F_1, \quad c_2 = \varepsilon\left(1 + \frac{I_p\alpha_\rho}{2} + d_1S_{00}\right).
\]

\[
S_{00} = \frac{\alpha_1}{\alpha_2d_1}\left(\frac{x_r - 1}{2}I_p - 1\right) - \frac{1}{d_1}\left(1 + \frac{I_p\alpha_\rho}{2}\right), \quad F_1 = \frac{\left(\frac{x_r - 1}{2}I_p - 1\right)}{\left(1 + \frac{I_p\alpha_\rho}{2} + d_1S_{00}\right)}.
\]  

Here, \( a = 1 \) for \( S_{30} = S_{00} \) (right circularly polarised SOP) and \( a = -1 \) for \( S_{30} = -S_{00} \) (left circularly polarised SOP). The results of the eigenvalue analysis are shown in figure 9.11(a)–(c).

As follows from figure 9.11(a), the position of satellites located nearby (~10^{-2}f) of the main peak in the RF spectrum with \( q = 0 \) (branch II) are independent of the in-cavity polarisation controller tuning the same as shown in figure 9.8(b). As follows from figure 9.11(b) for \( \rho = 0, \text{Re}(\lambda_q = 1) > \text{Re}(\lambda_q = 2) > \text{Re}(\lambda_q = 3) \), i.e. unlike the previous case shown in figure 9.10, the vector MMI results in the excitation of few longitudinal modes. Tuning the POC2 leads to increased \( \rho \) and thus results in increasing the maximum of the \( \text{Re}(\lambda_q = 1) \), which is proportional to the amplitude of the oscillations (figure 9.11(b)). Though \( \text{Re}(\beta) < 0 \) for the branch II, the satellites can be excited through parametric phase locking (PPHL) with the frequencies of branch
III and IV. The PPHL results in increased spectral power in the RF spectrum, shown in figure 9.8(b), caused by the adjustment of the angle $\rho$ leading to the mismatch between the frequencies from the branches I and III/IV proportional to the frequency from branch II (figure 9.11(c)). In view of the fact that the frequency's splitting is proportional to the angle $\rho$ (figure 9.11(c)), harmonic oscillations appear as shown in figure 9.8(a). Coupling between left- and right-circularly polarised SOPs results in switching between SOPs having different eigenfrequencies, as shown in figure 9.11(c). By adjusting $\rho$, a rational relation between SOP frequencies can be realised, which results in quasiperiodic motion (torus in the phase space) and further to chaotic behaviour through the Ruelle–Takens–Newhouse scenario (route to chaos via two-dimensional torus destruction [54]).

9.4 Summary

To summarise, we reviewed our recent experimental and theoretical data, demonstrating a new type of vector rogue wave emerging at a slow timescale of tens of thousands of roundtrips and resulting from the interaction between the orthogonal SOPs in an Er-doped mode-locked fibre laser with a saturable absorber. By tuning the in-cavity and pump wave polarisation controllers, we adjust the coupling between SOPs and thus provide conditions for the appearance of polarisation instabilities driven by slow bright and dark rogue waves. Such RWs complement fast RWs in the form of soliton rain visible in the timescale of a roundtrip. For a long fibre laser without a generic mode-locking mechanism, we reviewed our recent experimental and theoretical results on vector resonance multimode instability, which can result in vector resonance mode locking and complex multi-pulsing behaviour satisfying rogue wave criteria.

Annex I

*Experimental setup:* The laser (figure 9.1) comprises 1.1 m long erbium-doped fibre (EDF) with a nominal absorption ratio of 80 dB m$^{-1}$ at 1530 nm. The group velocity dispersion of the EDF is $+59$ ps$^2$ km$^{-1}$. A fibre pigtailed optical isolator (OISO) has been used to support a unidirectional operation of the laser. A 975 nm laser diode (LD) with maximum output power of 200 mW is used to pump the laser via a 980/1550 wavelength division multiplexer (WDM). A standard 70:30 fused coupler (OUTPUT C) has been installed to redirect 30% of the laser light out of the cavity. The rest of the laser cavity includes 1.22 m of OFS980 fibre and 4.4 m of SMF 28 fibre. The CNT mode-locker is used in the form of a piece of carbon nanotube polymer composite film placed between two standard fibre connectors supplemented by index matching for minimisation of insertion losses. The total cavity GVD is of $-0.04$ ps$^2$ nm$^{-1}$. Two polarisation controllers (POC1 and POC2) are installed to adjust the SOP of the pump wave and the in-cavity birefringence. A polarimeter (*Thorlabs IPM5300*, 1 $\mu$s resolution, 1024 samples) has been connected to the output. The laser signal is measured using a UDP–15–IR–2FC detector with a bandwidth of 17 GHz; the electric signal from the detector has been recorded by a *Tektronix DPO7254* oscilloscope (with 25 ps sample interval and 20 K in each
Figure A1. Optical spectrum of the laser emission: \( J_p = 140 \) mA (red) and 170 mA (blue).

Figure A2. Autocorrelation traces for \( J_p = 140 \) mA (a); 170 mA (b).

Oscilloscope. POC1 and POC2 have been adjusted to find conditions for RW emergence. The optical spectrum has been measured with the optical spectrum analyser (ANDO AQ6317B) and the pulse width, with the help of the autocorrelator (Pulsecheck). The oscilloscope data comprises 11 oscilloscopes of the pulse dynamics with 20 K points in each oscilloscope, the autocorrelation data consists of 8 traces, and polarimetry data includes 20 traces.

Optical spectrum: The dispersion map comprises anomalous dispersion of the EDF and normal dispersion of the SMF, and so the laser configuration is typical for stretched pulse laser as per the spectrum shown in figure A1 [55].

Autocorrelation traces: Detailed autocorrelation analysis shown in figure A2(a) reveals pulse width of about 700 fs and a fluctuating distance between the main pulse and satellites from 1 to 7 ps for \( J_p = 140 \) mA. The presence of soliton rain (figure A2(a), \( J_p = 140 \) mA), pedestal in the autocorrelation trace, and the smooth spectrum (figure A1) indicate the presence of loosely bound solitons moving in the range of delays from ps to ns with respect to the main pulse. With the increase of pump current to 170 mA (figure A2(b)), the satellites are located far from the main peak (> 5 ps), which means that loosely bound soliton interaction is weaker as compared to the previous case.
Numerical simulation (3D SOP rotation): In-cavity polarisation controller and a fibre patchcord connecting the laser output with the polarimeter provide 3D rotation (around axes $S_1$, $S_2$, $S_3$) of the initial SOP as follows [56]:

$$
\begin{pmatrix}
\tilde{s}_1 \\
\tilde{s}_2 \\
\tilde{s}_3
\end{pmatrix} =
\begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
0 & 0 & 0
\end{bmatrix}
\begin{pmatrix}
s_1 \\
s_2 \\
s_3
\end{pmatrix}.
$$

$$a_{11} = \cos(\psi) \cos(\gamma), \quad a_{12} = \cos(\gamma) \sin(\alpha) \sin(\psi) - \cos(\alpha) \sin(\gamma),$$

$$a_{13} = \cos(\alpha) \cos(\gamma) \sin(\psi) + \sin(\alpha) \sin(\gamma),$$

$$a_{21} = \cos(\psi) \sin(\gamma), \quad a_{22} = \cos(\alpha) \cos(\gamma) + \sin(\alpha) \sin(\psi) \sin(\gamma),$$

$$a_{23} = -\cos(\gamma) \sin(\alpha) + \sin(\psi) \sin(\gamma),$$

$$a_{31} = -\sin(\gamma), \quad a_{32} = \cos(\psi) \sin(\alpha), \quad a_{33} = \cos(\alpha) \cos(\psi).$$

Figure A3. Dynamics of the output powers at orthogonally polarised SOPs: $I_x$ (red), $I_y$ (blue), and $I = I_x + I_y$ (black). Parameters: (a) $\alpha = \psi = \gamma = 0$, (b) $\alpha = \pi 3$, $\psi = 0$, $\gamma = \pi 5$, (c) $\alpha = \pi 25$, $\psi = \pi 15$, $\gamma = \pi 10$; $I_x = 75$, $a_{11} = 21.53$, $a_{0} = 0.136$, $a_{+} = 1.8 \times 10^{-3}$, $a_{-} = 2.533$, $\chi = 2.3$, $\Delta = 0.1$, $I_y = 75$, $\gamma = 2 \times 10^{-3}$, $\epsilon = 10^{-3}$, $\delta = 0.85$ (elliptically polarised pump), $\beta = \pi \times 10^{-3}$.
Here, $\alpha$, $\psi$, and $\gamma$ are Euler angles [56]. Transformation of the output power dynamics for two orthogonal SOPs ($I_r$ and $I_s$) is shown in figure A3. As follows from figure A3(b), 3D rotations lead to a dynamic similar to the dynamics observed experimentally in figure 9.3(a1).

**Annex II**

**Experimental setup:** The schematic of the unidirectional cavity fibre laser is shown in figure A4. The cavity is comprised of 1 m of Er-doped fibre (Liekki Er80-8/125) and 614 m of single-mode fibre SMF-28. An optical isolator (OISO) with 51 dB back losses has been used to provide unidirectional lasing. The 80/20 fibre coupler was used to redirect the part of the signal outside the cavity. The cavity was pumped via a 1480/1550 WDM, using a 1480 nm laser diode (FOL14xx series) with an in-built isolator. The pump power was measured after the pump polarisation controller (POC1) and wavelength division multiplexing (WDM). The signal was detected using a photodetector with a bandwidth of 17 GHz (InGaAsUDP-15-IR-2 FC) connected to a 2.5 GHz sampling oscilloscope (Tektronix DPO7254) and electrical spectrum analyzer (ROHDE and SCHWARZ FSV Signal Analyzer, 10 Hz/13.6 GHz). An in-line polarimeter (Thorlabs IPM5300) was used to record the state and degree of polarisation (SOP and DOP), respectively.

**Linear stability analysis:** To find eigenvalues, we substitute the following ansatz into equation (9.3):

$$\mathbf{F}(t, z) \equiv \begin{bmatrix} S_0 \ S_1 \ S_2 \ S_3 \ f_2 \ f_3 \end{bmatrix}^T = \mathbf{F}_0 + [x_0 x_1 x_2 x_3 x_4 x_5] \exp(\lambda t + qz), \quad (A2)$$

where $\mathbf{F}_0 = (S_{00} \ S_{01} \ 0 \ 0 \ f_{10} \ f_{20})^T$. 

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As a result, we find the following equation for eigenvalues:

$$\det \begin{bmatrix}
  a_1 - iq - \lambda & 0 & 0 & 0 & a_2 & a_2 & 0 \\
  -a_1 & a_1 - iq - \lambda & 0 & 0 & a_2 & a_2 & 0 \\
  0 & 0 & a_1 - iq - \lambda & a_1 + a_4 & 0 & 0 & a_2 \\
  0 & 0 & -a_5 & a_1 - iq - \lambda & 0 & 0 & \Delta a_3 \\
  b_2 & b_4 & 0 & 0 & b_1 - \lambda & b_3 & 0 \\
  b_4 & b_2 & 0 & 0 & b_3 & b_1 - \lambda & 0 \\
  0 & 0 & \frac{b_2}{2} & 0 & 0 & 0 & b_1 - \lambda
\end{bmatrix} = 0, \quad (A3)$$

where

$$a_1 = \frac{2a_1}{1 + \Delta^2 f_{10} - a_2}, \quad a_2 = \frac{2a_1}{1 + \Delta^2 f_{00}}, \quad a_3 = \frac{2a_1 \Delta}{1 + \Delta^2 f_{20}} - 2\beta, \quad a_4 = -\gamma S_{00},$$

$$b_1 = -\epsilon \left(1 + \frac{I_\rho \xi}{2} + d_0 S_{00}\right), \quad b_2 = -\epsilon d_0 f_{10}, \quad b_3 = -\epsilon \left(d_0 S_0 + \frac{I_\rho \xi^2}{2}\right), \quad b_4 = -\epsilon d_0 f_{20}.$$

As a result, equation (A3) has three branches of eigenvalues:

$$(I) \quad \lambda - 2a_1 + 2iq = 0,$$

$$(II) \quad \lambda^3 + [-2b_1 + iq]\lambda^2 + \left(b_1^2 - \frac{b_1^2}{2} - \frac{3a_2 b_3}{2} - a_2 b_4 - 2iq b_1\right) \lambda + \frac{3a_2 b_1 b_2}{2} + 2a_2 b_1 b_4 = 0,$$

$$(III) \quad \lambda^2 + [-2a_1 - b_1 + 2iq]\lambda^2 + \left(a_1^2 + 2b_1 a_1 + a_3^2 + a_4 a_3 - q^2 - \frac{a_2 b_2^2}{2} - 2ib_1 q - 2ia q\right)$$

$$+ b_2 q^2 - a_1^2 b_1 - a_1^2 b_1 + \frac{a_1 a_2 b_2}{2} - a_2 a_1 b_1 - \frac{a_2 a_1 b_2 \Delta}{2} - \frac{a_2 a_1 b_3 \Delta}{2} - \frac{ia q b_2 q}{2} = 0.$$ 

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