QUANTUM EVOLUTION OF THE UNIVERSE IN THE CONSTRAINED QUASI-HEISENBERG PICTURE: FROM QUANTA TO CLASSICS?

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The quasi-Heisenberg picture of a minisuperspace model is considered. The suggested scheme consists in quantizing the equation of motion and interprets all observables including the Universe scale factor as time-dependent (quasi-Heisenberg) operators acting in the space of solutions of the Wheeler–DeWitt equation. The Klein-Gordon normalization of the wave function is used. An inflationary stage is considered numerically in the framework of the Wigner–Weyl phase-space formalism. For an inflationary model of “chaotic inflation” type it is found that the dispersion of the Universe scale factor grows during inflation.

1. Introduction

It is generally accepted that quantum effects must be taken into account at the initial stage of the cosmological evolution. Their description requires an appropriate quantization scheme. A variety of quantization schemes for minisuperspace models can be roughly divided into two classes: imposing the constraints i) “before quantization” [1, 2] and ii) “after quantization” [3, 4] (see also reviews comparing the two approaches [5, 6]). In the former, the constraints are used to exclude “non-physical” degrees of freedom, which allows constructing a Hamiltonian acting in the reduced “physical” phase space. In such models, the Universe dynamics is introduced by a time-dependent gauge. Gauges of this type should identify the Universe scale factor with a prescribed monotonic function of time [7, 8]. This results in a non-vanishing and generically non-stationary Hamiltonian of the system and, thus, in a time-dependent wave function. Such a procedure cannot be fully satisfactory, since it requires a priori introduction of an arbitrary function and does not allow considering the Universe scale factor as a quantum observable.

Alternative schemes prefer imposing the constraint “after quantization”, which leads to the Wheeler–DeWitt equation for quantum states of Universe [3, 4]. We believe that it is the most correct description of the quantum Universe. Nevertheless, there remains the problem of extracting information about the Universe time evolution, because there is no explicit “time” in the corresponding Wheeler–DeWitt equation. Possible solutions and interpretations of this problem like introducing time along quasi-classical trajectories, or subdividing the Universe into classical and quantum parts, have been offered [9].

However, let us note that in the Heisenberg picture the absence of evolution of mean values can be proved only for the Schrödinger normalization of the Universe wave function [10]. On the other hand, the Universe wave function cannot be normalized in the Schrödinger style if the most natural Laplacian ordering of the operators in the Wheeler–DeWitt equation is chosen (for a closed Universe and unnatural operator ordering, the Schrödinger norm can be obtained [11]). It gives a hope that some Heisenberg-like picture is possible if the wave function is normalized in the Klein–Gordon style. Certainly, the ordinary Heisenberg operators are not suitable for this aim because they are not Hermitian in the

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normalization above.

2. Quantization rules, operator equations of motion, mean values

Let us start with the Einstein action for gravity and a one-component real scalar field and restrict our consideration to the homogeneous, isotropic and flat metric

$$ ds^2 = N^2(t) dt^2 - a^2(t) dr^2. $$

Here the lapse function $N$ represents the general time coordinate transformation freedom. For the restricted metric, the total action becomes

$$ S = \int \left\{ -\frac{3}{8\pi G N} \frac{a^2}{N} + \frac{1}{2} a^2 \dot{\phi}^2 \right\} dt, $$

where $\Omega$ is a constant determining the volume of the Universe. It is equal to $2\pi^2$ for a closed Universe and should be properly fixed for the flat one (we choose $\Omega = 1$).

The action (2) can be obtained from the following expression by varying in $p_\phi$ and $p_\alpha$:

$$ S = \int \left\{ p_\phi \dot{\phi} - p_\alpha \dot{\alpha} + N \left( \frac{2\pi G a^2}{3a} - \frac{p_\phi^2}{2a^3} - a^3 V(\phi) \right) \right\} dt. $$

Varying in $N$ gives the constraint

$$ H = -\frac{2\pi G a^2}{3a} + \frac{p_\phi^2}{2a^3} + a^3 V(\phi) = 0. $$

Let us first consider a universe with $V(\phi) = 0$, corresponding to the Hamiltonian (in the units $4\pi G/3 = 1$)

$$ H_0 = -\frac{p_\phi^2}{2a^3} + \frac{p_\phi^2}{2a^3}. $$

After quantization, this constraint turns into the Wheeler–DeWitt equation

$$ \hat{H}_0 \psi = \left( \frac{1}{2a^3} \frac{\partial}{\partial a} a \frac{\partial}{\partial a} - \frac{1}{2a^3} \frac{\partial^2}{\partial \phi^2} \right) \psi = 0. $$

An expression for the wave function satisfying (5) is

$$ \psi_k(\alpha, \phi) = a^{\pm |k|} e^{ik\phi}. $$

Precisely as in the case of the Klein–Gordon equation, we should choose only positive-frequency solutions [9]. Thus, the wave packet

$$ \psi(\alpha, \phi) = \int c(k) \frac{a^{-i|k|}}{\sqrt{4\pi|k|}} e^{i k \phi} dk $$

will be normalized by

$$ i a \int \left( \frac{\partial \psi}{\partial \alpha} \psi^* - \frac{\partial \psi^*}{\partial \alpha} \psi \right) d\phi = \int c^*(k)c(k) dk = 1, $$

where some hyperplane $a = \text{const}$ is chosen.

Classical equations of motion

$$ p_\phi(t) = 0, \quad (a^3(t))' = 3p_\alpha, \quad (p_\alpha) = -3H_0 $$

can be obtained from the classical Hamiltonian $H_0$ by taking Poisson brackets.

To quantize the equations of motion, it is sufficient to specify commutation relations for the operators at an initial time instant $t = 0$. According to the Dirac quantization procedure [12], besides the hamiltonian constraint $\Phi_1 = -p_\phi^2/a + p_\phi^2/a^2$ (see (4)), we must set some additional gauge-fixing constraint, which can be chosen in our case as $\Phi_2 = a = \text{const}$ since the hyperplane $a = \text{const}$ is chosen earlier (see (7)) for normalization of the wave function in the Klein–Gordon style. In addition to the ordinary Poisson brackets, the Dirac brackets [12,13] should be introduced:

$$ \{A, B\}_D = \{A, B\} - \{A, \Phi_i\}(C^{-1})_{ij}\{\Phi_j, B\}, $$

where $C$ is a nonsingular matrix with the elements $C_{ij} = \{\Phi_i, \Phi_j\}$, and $C^{-1}$ is the inverse matrix. Quantization consists in postulating the commutators to be equal to the Dirac brackets with the variables replaced by operators:

$$ [\hat{\eta}, \hat{\eta}'] = -i(\eta, \eta')_D \mid_{\eta = \eta'}. $$

Here $\eta$ implies the set of the canonical variables $p_\alpha, x^\nu$.

In contrast to the usual formalism of Refs. [1,8,13,14], we postulate to impose the constraints $\Phi_1 = 0$ and $\Phi_2 = 0$ at the only hyperplane $t = 0$. Consequently, the quasi-Heisenberg operators obey the commutation relations obtained from the Dirac quantization procedure at the initial instant $t = 0$. Direct evaluation gives

$$ [\hat{p}_\alpha(0), \hat{a}(0)] = 0, \quad [\hat{p}_\phi(0), \hat{a}(0)] = 0, $$
$$ [\hat{p}_\phi(0), \hat{\phi}(0)] = -i, $$
$$ [\hat{p}_\alpha(0), \hat{\phi}(0)] = -i \frac{\hat{p}_\phi(0)}{\hat{p}_\phi(0)\hat{a}^2(0)}. $$

One must solve Eqs. (8) with given initial commutation relations. In contract to the ordinary Heisenberg operators, the quasi-Heisenberg operators do not conserve their commutation relations during the evolution\(^3\). The commutation relations (11) can be satisfied by

$$ \hat{a}(0) = \text{const} = a, \quad \hat{p}_\phi(0) = \hat{\phi}_\phi, \quad \hat{p}_\alpha(0) = j\hat{\phi}_\phi/a, \quad \hat{\phi}(0) = \phi, $$

where $\hat{\phi}_\phi = -i \frac{\partial}{\partial \phi}$. The variable $a = \hat{a}(0)$ is a c-number now because it commutes with all operators [14]. Solutions to Eqs. (8) are

$$ \hat{\phi}(t) = \hat{\phi}(0) + \frac{a^3}{3\hat{p}_\phi} \ln(a^3 + 3|\hat{p}_\phi| t) - \frac{\hat{p}_\phi}{|\hat{p}_\phi|} \ln a. $$

\(^3\)This quantization scheme has many common features with the relativistic-particle-clock model (i.e., a particle having its own clock, for instance, a radioactive particle) [10]
We imply that these quasi-Heisenberg operators act in Hilbert space with the Klein-Gordon scalar product. An expression for the mean value of an observable is
\[
\langle \hat{A}(t) \rangle = \int \psi^* (a, \phi) D^{1/4} \hat{A}(t) D^{-1/4} \frac{\partial}{\partial a} \psi(a, \phi) \, da + \int \psi^* (a, \phi) D^{1/4} \hat{A}(t) D^{-1/4} \psi(a, \phi) \, da \sim 0 ,
\]
where the operator \( D = -\partial^2 / \partial \phi^2 + 2a^2 V(\phi) \) (since, as \( a \to 0 \), the \( V(\phi) \) term can be omitted in the expression for \( D \)). Eq. (14) is a particular case of the equation suggested in Ref. [15], where a one-particle picture of the Klein-Gordon equation in the Foldy-Wouthuysen representation was considered. The adequacy of this definition can be seen in the momentum representation of the \( \phi \) variable: \( \hat{p}_\phi = k \) and \( \hat{A} = i \partial / \partial k \), where Eq. (14) gives
\[
\langle \hat{A}(t) \rangle = \int a^{ilk} c^*(k) \hat{A}(t, \hat{\phi}, k, a) a^{-ilk} c(k) \, dk \bigg|_{a \to 0} ,
\]
which is similar to the ordinary quantum-mechanical definition and certainly possesses hermiticity.

Evaluation of the mean value of the scalar field \( \phi(t) \) and the mean value of \( \hat{A}(t) \) over the wave packet (6) leads to
\[
\langle A(t) \rangle = 3t \int |k(c(k))|^2 \, dk ,
\]
\[
\langle \phi(t) \rangle = \int \left( \frac{k}{3|k|} \ln(|k|) |c(k)|^2 + c^*(k) \frac{i}{\partial k} c(k) \right) \, dk .
\]

An infinity can be found in the last equation: for \( c(k) \), which does not tend to zero at small \( k \), the mean value of \( \phi(t) \) diverges. This is a manifestation of the well-known infrared divergency of a scalar field minimally coupled to gravity. Thus, not all possible \( c(k) \) are suitable for construction of the wave packets.

Let us consider the Hamiltonian \( H = H_0 + a^2 V_0 \), containing the cosmological constant \( V_0 \). An explicit solution for the wave function \( H \psi = 0 \) has the form
\[
\psi_k(a, \phi) = \left( \frac{8}{V_0} \right)^{1/6} \Gamma \left( 1 - \frac{i|k|}{3} \right) J_{\frac{i|k|}{3}} \left( \sqrt{\frac{2V_0}{3} a^3} \right) e^{ik\phi} ,
\]
where \( \Gamma(z) \) is the gamma function and \( J_\mu(z) \) is the Bessel function. The above wave function tends to \( a^{-i|k|} e^{ik\phi} \) asymptotically as \( a \to 0 \). Then, for finding the mean values, we can always build a wave packet from solutions of the free Wheeler–DeWitt equation \( a^{-i|k|} e^{ik\phi} \). This reasoning holds for any potential \( V(\phi) \) because it contributes into the Hamiltonian as a term multiplied by \( a^2 \).

3. Operator equations for a quadratic inflationary potential and Wigner-Weyl evolution of minisuperspace

As has been discussed, the quantization procedure consists in quantizing the equations of motion, i.e., consid-

![Figure 1: Contour plots of Wigner functions of the Universe, numerically obtained mean values of \( \hat{A}(t) \), \( \phi(t) \), and their dispersions for the wave packets \( c(k) = e^{c \left( \frac{2}{3} \right)} \exp(i\phi_0 - k^2 - 1/k^2) \) (left panel) and \( c(k) = 2\sqrt{5} e^{\alpha\sqrt{6} \left( \frac{2}{3} \right) \exp(-600k^2 - 1/k^2) \) (right panel).]

ering them as operator equations. These equations must be solved with initial conditions obeying a constraint at \( t = 0 \). For the Hamiltonian
\[
H = -\frac{p_\phi^2}{2a} + \frac{p_\phi^2}{2a^2} + a^2 \frac{m^2 \phi^2}{2} \tag{15}
\]
we have equations of motion for the logarithm of the Universe scale factor \( \alpha \equiv \ln \alpha \) and the scalar field amplitude:
\[
\dot{\alpha} + \frac{3}{2} \alpha^2 - \frac{3}{2} m^2 \dot{\phi}^2 + \frac{3}{2} \phi^2 = 0 ,
\]
\[
\dot{\phi} + \frac{3}{2} (\dot{\alpha} \dot{\phi} + \dot{\alpha} \dot{\phi}) + m^2 \phi = 0 ,
\]
where symmetric ordering is used.

A numerical consideration of these operator equations can be realized in the framework of the Weyl-Wigner phase-space formalism [16], which associates every operator acting on \( \phi \) variable with a Weyl symbol: \( W[A] = A(k, \phi) \). The simplest Weyl symbols in our case are: \( W[-i\delta_{k,k}] = k \), \( W[\phi] = \phi \) and the Weyl symbol of the symmetrized product of operators reads
\[
W \left[ \frac{1}{2} (\hat{A} \hat{B} + \hat{B} \hat{A}) \right] = \cos \left( \frac{h}{2} \frac{\partial}{\partial \phi_1} \frac{\partial}{\partial k_2} \right) A(k_1, \phi_1) B(k_2, \phi_2) \bigg|_{k_1 = k_2 = k} \tag{17}
\]
Here the Planck constant \( h = 1 \) is restored only to point the order of the cosine expansion in the subsequent numerical calculations. In the numerical consideration we
have solved the set of partial differential equations [10] obtained from (16) by expanding products of operators to the second order in $\hbar$.

The state of the Universe is described by the Wigner function $\rho(k, \phi)$, which is constructed on the basis of the definition (14) and is given by

$$\rho(k, \phi) = \frac{1}{\pi} \int c^*(2k-q)c(q)\exp[-i|q|+i|2k-q|]e^{2i(q-k)\phi}dq$$

in the momentum representation of the wave function corresponding to Eq. (6).

Linde's "chaotic inflation" [17] supposes that the value of the potential at an initial stage of inflation must be of the order of Planck's mass $M_p$ in the fourth power, $V^{1/4} \sim M_p$, where $M_p = G^{-1/2} = \sqrt{4\pi/3}$ (in our units). Hence, the corresponding value of the scalar field is $\phi_0 = \sqrt{2}M_p^2/m$. The constant $m^2$ dictated by the COBE data is

$$m^2 \sim 10^{-12}M_p^2 = \frac{4\pi}{3}10^{-12}.$$  

Still for the purposes of visuality of the numerical calculations we take $m^2 = 1.7 \times 10^{-3}$ to reflect the fact that $m^2 \ll M_p^2$, and the initial scalar field is sufficiently large. There are two possibilities of creating a large scalar field: the first one is a wave packet with a non-zero mean field $\phi_0$; for instance,

$$c(k) = e^{i2(2/\pi)^{1/4}}\exp(ik\phi_0 - k^2 - 1/k^2).$$

The second one is a "squeezed" packet having a small uncertainty of $k$ but giving a large scalar field squared, for instance,

$$c(k) = 2\sqrt{5}e^{20\sqrt{6}(3i/\pi)^{1/4}}\exp(-600k^2 - 1/k^2).$$

For the both packets, the function $c(k)$ contains the factor $\exp(-1/k^2)$ suppressing the infrared divergence. The corresponding Wigner functions as well as the evolution of the operator expectation values and their dispersions are shown in Fig. 1.

The main conclusion is that dispersion of the logarithm of the Universe scalar scale does not vanish during inflation. Even for a wave packet having a large mean value of the scalar field $\phi_0$ and a small dispersion, the dispersion of the logarithm of the scalar field increases during inflation without decay after the end of inflation.

In our particular case (a small scalar field mass), the higher (in $\hbar$) order corrections in the cosine expansion (17) do not change the picture qualitatively. Smallness of the corrections indicates that accurate solutions of the operator equations of motion (16) for the particular set of parameters are obtained.

4. Conclusion

A quantization procedure for the equations of motion has been introduced, that results in quasi-Heisenberg operators which are Hermitian when the Universe wave function is normalized in the Klein-Gordon style.

For a quadratic inflationary potential, numerical calculations have demonstrated that the dispersion of the logarithm of the scale factor grows during inflation and approaches some constant at the end of inflation. An interpretation of this fact can be that, after inflation, the Universe scale factor is projected into different values in different spatial regions due to the "self-measurement" process associated with the appearance of local degrees of freedom during particle creation. The quantum dispersion of the scale factor turns into dispersion of the scale factor in different spatial regions and results in a highly non-uniform Universe. To reconcile this conclusion with the experimental data showing no significant dispersion of the Hubble constant measured in different directions, we may suggest that these spatial regions must have a super-Hubble size at present time. On the other hand, there exist other inflationary models leading to a negligible dispersion of the Universe scale factor after inflation [10].

References