Optics and Chaos

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Optics and Chaos: Chaotic, Rogue, and Noisy Optical Dissipative Solitons

27.1 Introduction

In the last decade, the concept of a dissipative soliton (DS), which is a strongly localized and stable coherent structure emergent in a nonlinear dissipative system far from the thermodynamic equilibrium, actively developed and became well established. This concept is highly useful in very different fields of science, ranging from field theory and cosmology, optics and condensed-matter physics, to biology and medicine [2,13,15,16,38,52,135,255]. One may paraphrase: It is “apparent that solitons are around us. In the true sense of the word they are absolutely everywhere” [14]. Nonequilibrium character of a system where a DS emerges, requires a well-organized energy exchange of DS with an environment. In turn, this energy flow forms a nontrivial internal structure of a soliton, which provides the energy redistribution inside it (e.g., see [13,15,257]). In this respect, a DS is a primitive analog of a cell.

In particular, a DS can have an inhomogeneous phase-(ϕ)-distribution so that \( Q = d^2 \phi / dt^2 \neq 0 \) (here, \( t \) is a coordinate along which a DS is localized). The last value is called a “chirp” and, correspondingly, a DS with a substantially large chirp was named as a “chirped DS” (CDS) [196]. The unique feature of CDS is its capability to accumulate an energy \( E \) without stability loss so that \( E \propto Q \) [105,114]. As a result,

Note that the used designations are model-dependent. For instance, \( E, \phi, \) and \( t \) are the pulse energy, phase, and local time for a mode-locked laser [13], but are the number of particles (mass of condensate), momentum (wave number), and transverse spatial coordinate for a Bose–Einstein (BE) condensate [136].
CDS is energy (or “mass”) scalable \([8,105,114,196]\). This phenomenon resembles a resonant enhancement of oscillations in environment-coupled systems so that it was proposed to name it as a "DS resonance" (DSR) \([40]\).

This capacity of a DS to accumulate an "energy" (or a "mass," see footnote above) is of interest for a lot of applications. For instance, it provides the energy scaling of ultrashort laser pulses \([110]\) so that more than 10 MW peak powers and, respectively, more than \(10^{14}\) W/cm\(^2\) intensities become reachable directly from a laser operating at over-MHz repetition rates \([174]\). Such and quite reachable higher pulse powers bring the high-field physics on table tops of a mid-level university lab \([98]\). In particular, high-energy ultrashort pulse lasers nowadays allow such experiments as direct gas ionization and high-harmonic generation, pump-probe diffraction experiments with electrons and production of nanometer-scale structures at the surface of transparent materials, characterization and control of electronic dynamics, a variety of biophotonic and biomedical applications, etc. \([159,211,214]\). Moreover, the over-MHz pulse repetition rates provide a signal rate improvement factor of \(10^3 - 10^4\) in comparison with that of classical chirped-pulse amplifiers \([98]\). As a result, the signal-to-noise ratio enhances essentially, as well. But, besides a direct scientific and practical interest, an "avatar" of DS in the form of an ultrashort laser pulse can be considered as a testbed for exploring the DSs in whole \([8]\). A rapid progress of modern femtosecond laser technology provides an ideal playground for such an exploration so that the theoretical insights promise to become directly testable and, on the other part, the theory can be urged by new experimental challenges.

As a result of its nonequilibrium character, a DS can demonstrate a highly nontrivial dynamics, including the formation of multi-DS complexes \([227]\), DS explosions \([50]\), noise-like DSs \([101]\), etc. The resulting structures can be very complicated and consist of strongly or weakly interacting solitons (the so-called soliton molecules and gas) \([263]\) as well as the short-range noise-like oscillations inside a larger wave packet \([143]\) or the clusters of strongly interacting dark and gray solitons against a background ("condensate") \([238]\). The nonlinear dynamics of these structures can cause regular, chaotic-like, and turbulent behavior \([103]\). Such a behavior can be affected by excitation of internal perturbation modes arising from a nontrivial internal structure of a DS \([116]\). A perturbed DS resembles a glass of boiling water and it becomes very sensitive to noise (including quantum) influence. This sensitivity of a DS to a noise turns it into a "mesoscopic" quantum object \([253]\), whose properties remain unexplored.

Further contents can be outlined in the following way: (i) a digest of DS theory will be exposed, (ii) different scenarios of DS chaotization will be considered, (iii) noisy and rogue DS will be surveyed, and (iv) some quantum aspects of the soliton theory will be treated.

### 27.2 Concept of Optical Dissipative Soliton: Physical and Mathematical Aspects

There is a vast amount of literature regarding the theory of DSs. Some preliminary systematization can be found in References 2, 8, 13–16, 105, 110, 201, 204, 236, and 250. However, it is necessary at first to declare the stumbling block of this theory: absence of a unified viewpoint. There exist unbroken walls between the circles of the scientific community exploiting and exploring the concept of a DS: walls between the solid-state and fiber laser representations of the theory, condensed-matter physics, numerical and analytical approaches, solitonic and statistical concepts, etc.

Briefly and conditionally, the theoretical approaches to DS analysis can be divided into (1) numerical, (2) exact analytical, and (3) approximated analytical. The third approach includes the models based on (1) perturbative and (2) adiabatic models (AM) as well as models based on (3) phase-space truncation (i.e., variational approximation [VA] and method of moments [MM])\(^1\).

As was emphasized, linear and nonlinear dissipation is crucial for DS formation. The simplest and most studied models for such a type of phenomena are based on the different versions of the nonlinear complex

\(^{1}\) The statistical approaches will be only briefly outlined in this chapter.
Ginzburg–Landau equation (NCGLE) (e.g., see [15,21]). With regard to the physics of mode-locked lasers, the cubic NCGLE [94] is known as the Hauss master equation. Cubic-quintic as well as nonpolynomial nonlinear extensions of this equation, as was proved, are precise models for description of laser CDSs with different types of mode-locking [15,16,114,147,148,205,261] and are valid even in the case of lumped dynamics [55,261].

The generalized (1 + 1)-dimensional NCGLE can be expressed in the following form [110]:

\[
\frac{\partial a(z,t)}{\partial z} = \left[ -\sigma + a \frac{\partial^2}{\partial t^2} + g \langle |a|^2 \rangle \right] a(z,t) + i \left( \beta \frac{\partial^2}{\partial t^2} - \gamma |a|^2 - \chi |a|^4 \right) a(z,t). \tag{27.1}
\]

Here, for an optical field envelope \(a(z,t)\), the slowly varying envelope approximation [184] is assumed, which is valid until \(T \gg 1/\omega_0\) and \(\Delta \ll \omega_0\) (\(\omega_0\) is the optical wave carrier frequency, \(\Delta\) is the spectral half-width of DS, and \(T\) is its width). \(z\) is a “propagation distance” (or a time for a Bose–Einstein condensate). In a laser, a DS transits periodically the same elements during its evolution and this period is termed a laser cavity period \(T_{\text{rep}}\). For solid-state lasers as a rule, the DS dynamics is slow so that the corresponding evolution scale exceeds \(T_{\text{rep}}\). In this case, \(z\) can be scaled on a cavity length and interpreted as a “cavity round-trip number” \(N\). In a fiber laser, all parameters of Equation 27.1 is \(z\)-dependent and, thereby, the field evolution within one cavity round-trip has to be taken into consideration, as a rule.

The energy exchange with an environment can be characterized by a “net-loss” coefficient \(\sigma\):

\[
\frac{\partial a}{\partial z} = -\sigma(E)a = -\ell a + a \times \left[ \frac{g_0}{1 + E/E_S} \right] \frac{1}{\sqrt{1 + E/E_S}} , \tag{27.2}
\]

where \(g_0\) is the unsaturated gain defined by a pump (i.e., the gain coefficient for a small signal), \(E = \int_0^{T_{\text{rep}}} |a|^2 dt\) is the intracavity field energy (\(|a|^2\) has a dimension of power), and \(E_S = \hbar \omega_0 S/\sigma_g\) is the gain saturation energy (\(\sigma_g\) is the gain cross section and \(S\) is the laser beam area). Multipulse propagation of the pulse through an active medium during one cavity round-trip, as it takes a place in a thin-disk oscillator, has to be taken into account by a corresponding multiplier before \(E\) in Equation 27.2. The power-independent (“linear”) loss coefficient is \(\ell\). The “gain saturation” law is model-dependent (multipliers after brace in Equation 27.2 represent only two possible ones). If a DS is considered in the vicinity of a marginal stability threshold (where \(\sigma = 0\) by definition), one may expand \(\sigma\):

\[
\sigma(E) \approx \delta (E - E^*), \tag{27.3}
\]

where \(E^*\) is the energy of a marginally stable state corresponding to \(\sigma = 0\), and \(\delta \equiv (d\sigma/dE)|_{E=E^*}\). In a high-energy regime, the time dependence of \(E\) (or, in other words, the dynamic gain saturation) begins to play an important role and has to be described by a separate differential equation [110].

The \(\alpha\)-term in Equation 27.1 describes a frequency dissipation in a system (in a Bose–Einstein condensate, this term corresponds to a particle velocity dependence of a “leak” from the condensate). If a spectral filter has a profile \(\Phi(\omega)\), then its contribution can be described as

\[
\frac{\partial a(z,\omega)}{\partial z} = \Phi(\omega)a(z,\omega), \tag{27.4}
\]

where \(a(z,\omega)\) is the Fourier image of \(a(z,\tau)\).

* It can be very complicated for a fiber laser and is beyond the scope of a simple scalar model (27.1) [215].
Usually, the Lorenzian shape for $\Phi(\omega)$ is assumed:

$$\Phi(\omega) = \frac{g(\omega_0)}{1 + i \omega/\Omega_g}, \quad (27.5)$$

where $g(\omega_0)$ is a gain coefficient or a maximum transmission coefficient (e.g., for an output mirror). It is assumed that spectral filtering is centered at $\omega_0$. $\Omega_g$ is a filter (gain) bandwidth and then $\alpha \equiv g(\omega_0)/\Omega_g^2$ in Equation 27.1.

If $\Delta \ll \Omega_g$, one may expand Equation 27.5 and proceed in the time domain

$$\frac{\partial a(z, \tau)}{\partial z} \approx g(\omega_0) \left[ 1 - \frac{1}{\Omega_g} \frac{\partial}{\partial \tau} + \frac{1}{\Omega_g^2} \frac{\partial^2}{\partial \tau^2} - h. o. t. \right] a(z, \tau), \quad (27.6)$$

where $h.o.t.$ means the higher in $\partial/\partial\tau$-order terms, which are negligible as a rule (for some important exceptions, see, e.g., [15,55]).

The $\mathcal{H}(|a|^2)$-operator describes a so-called self-amplitude modulation (SAM) or a nonlinear gain. In a laser, this term corresponds to some power-discrimination process in a system. Such a process provides an effective gain growth with the power (at least up to some power level), which forms and stabilizes a DS. The mechanisms of SAM are various [188,248], but there are two simplest expressions for the $\mathcal{H}(|a|^2)$-operator, which are used extensively:

$$\mathcal{H}(|a|^2) = \begin{cases} 1 + \kappa |a|^2 & \text{if } \kappa |a|^2 > 1 - \zeta |a|^2 \\ \kappa |a|^2 (1 - \zeta |a|^2) & \text{if } \kappa |a|^2 \leq 1 - \zeta |a|^2 \end{cases} \quad (27.7)$$

The first SAM law describes a monotonic decrease (up to zero) of loss ($\mu$ is a loss level, or a so-called “loss modulation depth”) with the power $|a|^2$. The corresponding nonlinear coefficient $\kappa$ describes a “strength” of such a “loss saturation.” The second SAM law corresponds to a so-called cubic-quintic NCGLE [15] and describes a loss decrease with the power only up to some critical power level ($P_c = 1/2\kappa$) following which a loss begins to increase. The corresponding parameter of “SAM saturation” is defined by $\zeta$. In a BE condensate, the SAM can be connected with the multiparticle interactions resulting in the creation/annihilation of bosons.

The terms within square brackets in Equation 27.1 describe the dissipative factors in a system. The terms within braces are more convenient and correspond to the so-called cubic-quintic nonlinear Schrödinger equation, which describes dispersive (nondissipative) solitons in fiber optics, BE condensate, etc. (e.g., see [7,13,52,185,255]). The remarkable property of cubic nonlinear nondissipative reduction of Equation 27.1 is that it is completely integrable (e.g., see [48,79,170]) and, thereby, is completely traceable mathematically and, to a certain extent, is “linear-like.” A “true” (in the sense of “integrability”) soliton of such equation corresponds to the case of $\chi = 0, \beta < 0$ (so-called “anomalous” group-delay dispersion [GDD]) and $\gamma > 0$ (“self-focusing” nonlinearity or attractive three-bosonic potential).\footnote{It should be noted that $\mu$ has to be included in the definition of $\ell$ for such a form of SAM law.}

The imaginary nondissipative linear (“kinetic energy”) term in Equation 27.1 describes a GDD [188,248], which is the frequency dependence of the wave-packet propagation constant. Note that the dispersion coefficient $\beta$ is frequency-dependent, as well. In the simplest case, such a dependence can be expressed by the inclusion of higher-order derivatives into Equation 27.1 [7]. The corresponding terms are called the higher-order dispersions (HODs). As will shown below, the HOD plays an important role in the chaoticization of DS dynamics.

The imaginary nonlinear term in Equation 27.1 describes a self-phase modulation (SPM) [188,248], which is the power dependence of the wave-packet phase. As a rule, the cubic term (the corresponding “three-boson interaction” coefficient is $\gamma$) is sufficient to take into account the SPM, but the higher-order nonlinear corrections (e.g., $\chi$-coefficient in Equation 27.1) can affect the DS properties substantially [13,\footnote{The situation is completely reversible: a “true” soliton exists for $\beta > 0$ and $\gamma < 0$, as well.}
107]. In particular, a strongly “chirped” and, thereby, energy-scalable DS can develop under condition of “anomalous” GDD, when \( \gamma > 0 \) and \( \chi \neq 0 \) \([40, 41, 107, 158]\).

### 27.2.1 Numerical Study of DSs

Extensive numerical study of DSs of the \((1+1)\)-dimensional cubic-quintic NCGLF has been carried out by N.N. Akhmediev with coauthors [8, 13–16, 18, 39, 41, 80, 226]. The simulations have allowed finding the DS stability regions for some two-dimensional projections of NCGLF parametrical space. The summarizing description of the results obtained is presented in [110].

The most impressive results are

i. Parametric space of DS can have a reduced dimensionality, the corresponding phenomenon was called the dissipative soliton resonance (DSR).

The last term is sound because this phenomenon resembles a resonant enhancement of oscillations in environment-coupled systems. As a result, the DS becomes energy-scalable and can gain the energy without stability loss. This property is illustrated by Figure 27.1, where one can see a swift growth of DS energy along with the dispersion \( \beta \).

ii. The next important result is that DSR remains in models with a lumped evolution of DS.

Such models describe a DS evolution in a typical fiber laser, where the DS parameters can change dramatically during the propagation. The existence of DSR in this case is good news from the point of view of soliton energy scalability.

iii. DSR exists in both normal \((\beta > 0)\) and anomalous \((\beta < 0)\) regions of GDD.

The calculations demonstrate that the DS scaling properties for an anomalous GDD surpass those for a normal one. An “unusual” (due to the existence of a “true” soliton for \( \beta < 0 \)) anomalous dispersion regime of DS generation has been observed experimentally [62]. Note, that \( \chi \neq 0 \) is required for such a regime.

In spite of lasting achievements of numerical approaches in the DS exploration, they have some inherent shortcomings: (i) the parametrical space under consideration is not physically relevant owing to the dropping of the gain saturation, (ii) the true dimensionality of DS parametric space is not identified exactly.

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and (iii) the polarization dynamics does not get taken into account [215]. As has been shown, the inclusion of gain saturation allows a real-world description of the lasers producing DSs as well as it demonstrates a modification of the DSR conditions (e.g., see [43,55,125]). But the patterns revealed cannot be formulated as the quantitative and well-founded laws so far. It is clear that the only advanced and self-consistent analytical theory of DS would provide, in particular, a true representation of dissipative-solitonic parametric space and DSR conditions. A vector extension of the DS model (i.e., taking into account a polarization dynamics) offers supplementary issues as a result of doubling of a phase space dimensionality and taking into account the vector character of field–matter coupling [215].

27.2.2 Analytical Theories of DSs

Below, basic analytical models of DS will be briefly surveyed. For convenience, we divide these models into three groups based on (i) exact DS solution of Equation 27.1, (ii) solutions obtained from the so-called adiabatic approximation, and (iii) approximated solutions obtained from the variational method and the method of moments.

27.2.2.1 Exact DS Solutions of NCGLE

Since Equation 27.1 is nonintegrable, its exact soliton solutions are known only for a few cases. For instance, the cubic-quintic NCGLE in the cubic limit (\( \zeta = \chi = 0 \)) has an exact DS solution in the following form [95,166]:

\[
a(z, t) = a_0 \cosh^{-1}\psi \left( \frac{t}{T} \right) \exp(\imath q z),
\]

(27.8)

where \( a_0 \) is a soliton amplitude, \( T \) is its width, \( q \) is a soliton “wave number,” and \( \psi \) is a so-called chirp, which is a measure of soliton phase inhomogeneity (\( \psi \propto QT^2 \), see Section 27.1) or its spectral extra-broadening: \( \Delta \propto \sqrt{1+\psi^2/T} \) [95] (\( \Delta \) is a DS spectral half-width). These parameters can be expressed through the parameters of Equation 27.1. There are two disconnected regions of DS existence (such regions are disconnected by virtue of stability condition \( \sigma > 0 \), which can get broken in the vicinity of \( \beta = 0 \)): (i) anomalous dispersion regime with \( \psi \approx 0 \) and (ii) normal dispersion regime with \( \psi \neq 0 \), which corresponds to a CDS [105,110].

Solution (27.8) corresponding to a weakly nonlinear limit of NCGLE does not provide with an in-depth analysis of energy scalable DS. In particular, it is obvious that an absence of the SAM saturation (\( \zeta = 0 \)) enhances the tendency to a collapse-like instability of DS [42]. The general DS solutions of strongly nonlinear versions of Equation 27.1 (e.g., cubic-quintic NCGLE) are unknown. One may hope that they can be revealed on the basis of the algebraic nonperturbative techniques [48,79], which, nevertheless, are not developed sufficiently still. However, one known exact solution of the cubic-quintic NCGLE can provide with some insight into properties of DSs [18,41,80,200,226,240]:

\[
a(z, t) = \sqrt{\frac{A}{B + \cosh (t/T)}} \exp \left[ \frac{\imath q}{2} \ln \left( B + \cosh \left( \frac{t}{T} \right) \right) + \imath q z \right]
\]

(27.9)

where \( A, B, T, \psi, \) and \( q \) are the real constants characterizing pulse amplitude, shape, width, chirp, and wave number, respectively. It is important to emphasize that this partial solution exists for only certain algebraic relations imposed on the parameters of Equation 27.1 (i.e., this solution has a so-called “co-dimension one” [236]).

In spite of such a very specific property of Equation 27.9, it allows classifying systematically the DS spectra [200]. The crucial shortcoming of Equation 27.9 is that the strict restrictions are imposed on the NCGLE parameters. As a result, the DS cannot be traced within a broad multidimensional parametric range and the picture obtained is rather sporadic and is of interest only in the close relation with numerical results. Some additional information can be obtained on the basis of perturbation theory [140,164]. This
approach provides with a quite accurate approximation for a low-energy CDS, when $\sigma_\xi/\kappa \ll 1$ [107]. The corresponding solution is continuously extendible to Equation 27.8, when $\xi$ and $\chi \rightarrow 0$.

As further steps in the development of analytical techniques are required for the DS exploration, two powerful approximate methods have been proposed.

### 27.2.2.2 Adiabatic Theory of DS

Adiabatic theory of DS has been proposed in References 3 and 196 and can be put in a nutshell in the following way [110]. The theory is developed in three main steps: (i) the condition of $T \gg \sqrt{\beta}$ allows the adiabatic approximation for Equation 27.1; (ii) regularization (“renormalization”) procedure is applied to an expression for the soliton frequency deviation $\Omega = d\phi/dt$ (i.e., $Q = d\Omega/dt$), which excludes nonphysical solutions; and (iii) the method of stationary phase for a Fourier image of soliton complex envelope $a(z,\omega)$ is applied, which gives expressions for the soliton spectrum and its energy. The last step requires $|\psi|_{0} \gg 1$, i.e., $\beta \gg \alpha$ and $\gamma \gg \kappa$ in Equation 27.1.

This method has been applied for both versions of the SAM law (27.7) as well as for the case of higher-order SPM with $\chi \neq 0$ (see [105,110] for a more technical introduction). The adiabatic theory gives a set of interesting results providing with a deep insight into properties of CDSs.

i. There are two branches of CDSs, which correspond to different soliton energies for a fixed control parameter $C = \sigma_{\gamma}/\kappa$ or different control parameters for a fixed soliton energy.

Figure 27.2 demonstrates the typical power profiles $P(t) = |a(z, t)|^{2}$ and the corresponding frequency deviations $\Omega(t)$ for these two branches. The so-called positive branch of DS has higher energies and provides the energy scalability by means of pulse width scaling. A DS profile becomes flat-top in the process of such scaling.

ii. CDS spectra are truncated at some resonance frequency $\pm \Delta$ and have various profiles: flat-top, convex, concave, or concave-convex.

$\Delta$ (as well as $\omega$) has a sense of frequency measured from the soliton carrier frequency $\omega_{0}$. Figures 27.3 and 27.4 demonstrate some typical spectra of CDSs. The spectra can have strongly enhanced (“condensate”) spectral components at the edges (at $\pm \Delta$) (Figure 27.4). The nature of this phenomenon is closely connected with DS perturbation and chaoticization and will be considered below. When the energy scales and the DS soliton temporal profiles tend to a flat-top one, the spectral profile becomes bell-like (“Lorentzian”).

![Figure 27.2](image)

**Figure 27.2** Left: The positive-branch DS profiles $P(t)$ (solid curves) and frequency deviations $\Omega(t)$ (dashed curves) for the different $b = \gamma/\xi$. Right: Ditto for the negative-branch DS, $C = \sigma_{\gamma}/\kappa = 0$, $\sigma_{\gamma}/\kappa = 0.01$. (Adapted from V. L. Kalashnikov, Cism, 1:51–59, 1911.)

*The negative branch is quasi-(in the sense of possible stability loss) continuous connected with Equation 27.8.*
iii. The lasting achievement of adiabatic theory is a disclosure of the structure of DS parametric space and its relevant dimensionality. The corresponding representation of DS parametric space has been named as a "master diagram."

CDS "lives" inside a parametric space with reduced dimensionality (two-dimensional one for $\chi = 0$ and three-dimensional one for $\chi \neq 0$, see Equation 27.1). The structure of this space can be represented in the form of the so-called "master diagram" [124] and is completely characterized by a set of "isogains," that is, a set of curves, where a DS traces some fixed value of the $\sigma$-parameter (Equation 27.1). The marginal isogain curve $\sigma = 0$ corresponds to the DS stability border.

iv. Master diagram reveals the energy-scalability properties of DS.

As can be seen from Figure 27.5, the energy scalability of positive and negative branches of CDS differs: the negative branch requires a substantial decrease of $C$-parameter (e.g., as a result of the GDD growth) for energy scaling. The positive branch of DS excels the negative one in scalability. For instance, it has a "perfect" scalability for the cubic-quintic NCGLE (left side of Figure 27.5), which means that there exists
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\[ C = \frac{\sigma_0 \beta}{\kappa} \]

\[ E \equiv (2/\gamma \alpha)^{1/2} \]

\[ E \equiv (2/\gamma \alpha)^{1/2} \]

**FIGURE 27.5** Left: Master diagram for the CDS of cubic-quintic NGGLE, \( \chi = 0 \). There exists no soliton above the solid red curve (therein \( \sigma < 0 \), i.e., the vacuum of Equation 27.1 is unstable). Dashed curve divides the regions, where the positive- and negative-branches of CDS exist. Crosses (circles) correspond to the positive-(negative-)branch for an isosin \( \sigma_0/\kappa = 0.01 \). \( E \) is a DS energy. Right: Master diagram for the CDS corresponding to the upper expression for a SAM law in (27.7). Solid curve is the stability threshold. Dashed curve divides the regions of positive and negative branches of DS, which are shown for an isosin \( \sigma_0/\mu = 0.01 \) by crosses and circles, respectively. (Adapted from V. I. Kalashnikov, *Solid State Laser*, pages 145–184. InTech, 2012.)

\[ \lim_{\kappa \to 1/3} E = \infty \] along the marginal isosin. This phenomenon has been named the DSR (see above) and it means that the DS energy scaling for a fiber laser can be achieved by a plain laser lengthening.

As can be seen from the right picture in Figure 27.5, there exists no DSR, when the SAM is defined by the upper expression in Equation 27.7. But the positive branch of CDS is maximally scalable, as well. The corresponding asymptomatic energy scalability law can be expressed as \( E \approx 18\beta^2/\gamma \alpha^{3/2} \) \[ [110] \] .

v. Simple analytical expressions for the complex spectral amplitude of DS allow developing the perturbation theory in spectral domain.

Such a theory (e.g., see [105,108,130]) proved its usability in spectroscopy and promises a further progress in taking into consideration the higher-order dispersion terms in Equation 27.1.

One may conclude that the adiabatic theory provides with a deep insight into the physics of CDSs. The DS characteristics become easily traceable, and finding a true parametrical space of soliton allows looking at a DS from a unified point of view. Nevertheless, the underlying approximations of (i) strong domination of the nondissipative effects (GDD and SPM) over the dissipative ones (spectral filtering and SAM) and (ii) distributed character of a system impose some restrictions. There are another analytical approaches, which can shed light on some properties of a DS that go beyond the scope of the adiabatic theory.

### 27.2.2.3 Truncation of Phase Space: Variational Approximation and Method of Moments

This methods are based on a truncation of a space of (unknown) solutions of Equation 27.1 by means of its projection to a subspace of functions corresponding to soliton-like solutions. Such a truncation can be done by some appropriate ansatz into Equation 27.1 with the subsequent integral minimization procedure (for an overview, see [19,20,110,161,194,216,236]). The ansatz, as a rule, is the known analytical solution (27.9) or its representation (reduced and/or phase-modified). As a result, the complex dynamics of DS can be reduced to the comparatively simple one described by a set of ordinary differential equations governing an evolution of the ansatz parameters (pulse amplitude, width, chirp, etc.). Thereby, the problem becomes a semianalytical one [233,239].

The analysis, based on this approach, confirms a reduced dimensionality of DS parametric space. Moreover, in contrast with the adiabatic approximation, the analysis based on truncation of phase space allows
exploring a substantially broader range of parameters of Equation 27.1. In particular, this method works in both positive and negative dispersion ranges and allows taking into account a lumped evolution of DS and a DS polarization dynamics in a laser [24,25,27,28,54,56,236].

The example of master diagram obtained from the VA based on the ansatz (27.8) is shown in Figure 27.6. The right solid curve in Figure 27.6 corresponds to the stability threshold obtained from the adiabatic theory: the soliton is unstable on the right of this curve. The positive branch solution with convex spectrum, which is predicted by the adiabatic theory, cannot be obtained from Equation 27.8 so that the solution for the latter is situated on the right of the shown curves corresponding to the different values of \( \gamma / \kappa \) (Figure 27.6). These curves are the zero-level isogains (i.e., the curves of \( \sigma = 0 \)) for DS solutions of Equation 27.8. Note that the requirement of \( \gamma \gg \kappa \) is not essential for the VA. Nevertheless, one may see that all solutions have a single asymptotic (dashed curve) for \( C \ll 1 \) so that the master diagram is two-dimensional (i.e., it does not depend on the \( \gamma / \kappa \) value) in this limit. The asymptotical values of the pulse parameters along the dashed curve are [109]:

\[
E \approx \frac{17 \beta}{\sqrt{\alpha \kappa}}, \quad T \approx \frac{8 \gamma}{C \kappa \sqrt{\frac{\alpha \gamma}{\kappa}}},
\]

(27.10)

The importance of these scaling rules is that they correspond to the flat-top spectrum relevant to Equation 27.8, which has the spectral chirp with a weak frequency dependence within the range of \([\pm \Delta, \Delta]\). As a result, such a pulse is almost perfectly compressible (has a “maximal fidelity”), that is, the energy loss due to satellite generation in the process of CDS compression is minimal [267].

The prospects of further development of the method of phase space truncation can be shortly characterized as follows: (i) extension of the class of trial ansätze and advancement of the integration techniques aimed to an adequate description of a broader class of NCGLE; (ii) close cooperation with the adiabatic theory and the numerical simulations aimed to an adequate representation of the DS parametric space and obtaining of the energy scaling laws for the different types of NCGLE; (iii) most drastic advantage of

**FIGURE 27.6** Master diagram of CDS for the cubic-quintic NCGLE. The right solid curve corresponds to the stability threshold obtained from the adiabatic approximation. Another curves correspond to the thresholds obtained from the variational approximations with the ansatz (27.8) for the different values of \( \gamma / \kappa \). The dashed line corresponds to the asymptotic \( C \ll 1 \). (Adapted from V. L. Kalashnikov. Solid State Laser, pages 145–184. Intech, 2012.)
the considered method supposes the extension of NCGLE dimensionality (i.e., \([1 + 2]\) and \([1 + 3]\)), that is, the inclusion of the transverse spatial coordinates into consideration.\footnote{The corresponding DS solutions of NCGLE was named as spatiotemporal solitons, or "light bullets" [11,163,202,242,249].}

\subsection*{27.3 Chaotic and Rogue Dissipative Solitons}

Nondissipative ("true") solitons of integrable nonlinear systems do not demonstrate a chaotic behavior by definition. The situation changes radically for nonintegrable systems and, especially, for a DS. The latter corresponds to localized structure developing far from equilibrium in the systems with strong energy in/out flows. The scenarios of DS chaotization can be conditionally subdivided into three classes: (i) chaotic single soliton pulsations, (ii) structural soliton chaotization, and (iii) chaotization due to resonant interaction with linear waves.

\subsubsection*{27.3.1 Chaotic Single DS Pulsations}

The chaotic behavior of single DS can be considered in the framework of cubic-quintic CNGLE (27.1) and subdivided into two classes: (i) purely local dynamics with a time-independent \(\sigma\) (there is no dynamical gain saturation) and (ii) nonlocal one with a time-dependent \(\sigma\) (there is a dynamical gain saturation).

The first class of scenarios has been extensively analyzed numerically in References 12 and 224. The revealed types of chaotic DSs are diverse and will be briefly discussed below.

\subsubsection*{27.3.1.1 Exploding Solitons}

For this scenario, the DS envelope becomes increasingly "rippled" in the process of evolution [51,225] (see Figure 27.7). As a result, the soliton acquires a chaotic substructure and cracks into pieces. Then, this chaotic but localized structure "relaxes" and self-collects into a well-shaped DS again. The process repeats forever without a regular period.

The DS explosions can be explained in the following way [223]. DS can be unstable, relatively the slowly growing perturbations. When the amplitude of these perturbations becomes high, the nonlinearity mixes all perturbations that produce the radiative waves. As a result, the dynamics becomes chaotic but the radiative waves "cool" a system that restores a DS after this chaotic stage. Then, the process repeats. At certain values of parameters, the DS explosions can break the time symmetry of Equation 27.1 so that a DS begins to "wander" along the \(t\)-coordinate [223]. Such a "wandering" is also possible in the absence of soliton explosion and the corresponding DSs have been named as "creeping solitons" [12]. Their existence can be connected with the asymmetric "internal perturbational modes" of DS (see below).

In some way, the exploding DS belongs to the type of an irregularly repetitive structural soliton chaotization. The structural chaotization will be considered below. But another important feature of an exploding DS is a permanent presence of small (but periodically growing) internal perturbations. Such perturbations (or "internal perturbational modes" [141,193]) can result in the following chaotization scenario.

\subsubsection*{27.3.1.2 Internal Perturbational Modes of DS}

The interesting property of a DS is that it is not completely "solitary" and can have some nontrivial internal filling in the form of permanent nondestructive perturbations, or a solitonic "microflora." Such a "microflora" causes the DS pulsations (both regular and irregular, see Figure 27.8, left) so that the DS dynamics can resemble a strange attractor in low-dimensional systems [224].

The structure of this attractor can be reconstructed on the basis of standard approaches considered in Reference 1. The dimension of a phase space corresponding to a DS dynamics can be estimated on the basis of the "false nearest neighbors" method [1]. The task of this method is to exclude the intersections of trajectories in a phase space, which result from the projection of a true phase space into a low-dimensional one.
One can see from Figure 27.8 (right) that the phase space is three-dimensional for the GDD of 100 and 150 fs$^2$. The phase space reconstruction is shown in Figure 27.9, where the propagation lag $\tau = 35$ is defined from the first minimum of the autocorrelation function of the set of $\max(|a(z, t)|^2)$ [1]. The bifurcation resulting in such long-period pulsations (both regular and irregular) can be classified as the edge bifurcation [132, 133], which is caused by beatings between the perturbation modes. Such beatings are clearly visible in a DS spectrum evolution and resemble somewhat like the “boiling” of a localized spectrum profile (Figure 27.10).

As one can see from Figure 27.8 (left) and as was found in References 120 and 225, the regions of chaotic behavior can alternate with the region of regular DS pulsations and even coexist with both regularly pulsating and plain DSs. As was conjectured in Reference 116, the excitation of internal perturbational modes can prevent an energy scaling due to chaotization, even destruction of DS, and appearance of turbulence [213]. Also, the region of soliton chaotic pulsations can precede the region of multiple DS generation [26, 129] and, thereby, be closely connected with the phenomena of hysteresis and structural soliton chaotization.

### 27.3.2 Structural Soliton Chaotization and “Rogue” DSs

It was found [42] that an unsaturable SAM, i.e., a cubic nonlinear gain in Equation 27.1, causes a collapse instability of a DS. As a result, a DS becomes nose-like (or “noisy”) [101] that is, it transforms into a localized conglomerate of chaotically evolving spikes. Such local chaotization, which does not connect with a global gain evolution, can be classified as a local structural chaotization.

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*That is the $\sigma$-parameter is $t$-independent in Equation 27.1 and depends on only DS integral energy $E(z) = \int_0^{T_{rep}} |a(z, t)|^2 dt$ calculated over the current z-slice.*
27.3.2.1 Local Structural Chaotization

A structural chaotization can be characterized as a development of femtosecond-scale chaotical structure under a picosecond-scale averaged envelope [101,102,142,144,157,265,266]. One may conjecture that there are two aspects of such structural chaotization.

From the first point of view, a structurally chaotical DS can be treated as an irregularly evolving complex of strongly interacting solitons [129,146,160,264]. An example of such “soliton soup” (or “soliton gas”

[171,210]) is shown in Figure 27.11 (left). The chaotical self-emergence and self-annihilation of strongly interacting and bounded solitons can be interpreted as a result of nonresonant interaction with a “vacuum” of Equation 27.1 [109]. A vacuum becomes excited as a result of the growth of DS spectral loss [129] or switching of SAM from the positive to the negative passive feedback regime [142,144,157]. Note, that a “soliton gas” (or “solitonic turbulence” [257]) is dynamically nontrivial and can, in particular, evolve in the regular multisoliton complexes (soliton “molecules” or “photon condensate” [32], see Figure 27.11, right) with a possible soliton merging [260] and creation of “supersoliton,” which can be interpreted as a “rogue wave” (see below) or “wave-turbulence lasing” [32]. Another possible scenario is an interaction of localized “soliton soup” (“soliton cluster”) with a background in the form of the so-called “soliton rains” [44,45]: (i) “soliton cluster” permanently radiates the solitons or (ii) “cluster” permanently absorbs the solitons risen from a background.

From the second point of view, a structural chaotization may be connected with the strong four-wave mixing within a picosecond-scale localized DS (or soliton “condensed phase”). Such a mixing creates chaotic femtosecond-scale intensity beatings causing extremal spectrum broadening (in average); the process, which is akin to the picosecond or noise-driven (highly incoherent) supercontinuum generation [67,207]. In the last case, an extremely broadband radiation arises from an initially smooth pulse during its fragmentation and a subsequent power transfer between fragments seeded by noise perturbations [219]. This phenomenon, or generation of optical rogue waves, deserves special attention (see below).

Also, the structural chaotization can be interpreted from the point of view of “wave turbulence” [176], when a nonlinear four-wave mixing between the spectral components inside the DS spectrum “leads to a random energy transfer between waves and to enhancement of mode dephasing” [235]. In the time domain, such a turbulence corresponds to the laminar-turbulent transition from a “light condensate” (in the form of a continuous-wave, left picture in Figure 27.12 [238]; or a continuous wave creating a cluster of self-similar evolving DSs, right pictures in Figure 27.12 [113]) to a turbulent cluster of strongly

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* As a rule, the underlying equation for supercontinuum generation is the nonlinear Schrödinger equation supplemented with the HOD and stimulated Raman scattering (SRS) terms [64,218]. The incoherent picosecond supercontinuum generation is characterized by extremal shot-to-shot spectrum variations (see [63] and references therein).
interacting gray and dark solitons under a background corresponding to an “average” (continuous-wave or DS) envelope.

The interesting feature of DS spectrum following its energy scaling is an appearance of the so-called “spectral condensate” [235], when the main power is concentrated within narrow spectral region at the centrum of the spectrum (Figure 27.3, left; the so-called “finger-like” spectra [196]) or its edges (Figures 27.4 and 27.10 and [130]). A spectral condensation at the centrum of the spectrum can be described in the framework of the adiabatic theory of DS [110], which represents the DS spectrum as a truncated Lorentzian. When the DSR condition $C \rightarrow 2/3$ is satisfied, the truncation frequency $\Delta$ tends to the constant $\sqrt{\gamma/\beta}$, while the width of Lorentz spike at the spectrum center tends to zero. In the time domain, such a DS has a broad flat-top envelope. Thus, localization in the spectral domain entails delocalization in the time domain which, in the case of turbulent behavior, leads to a phenomenon of the so-called “spectral incoherent soliton” [172,254] when a coherent-like localization occurs in spectral (not time) domain. The condensation at the spectrum edges can be considered in the frameworks of perturbation theory predicting a resonant enhancement of perturbations with the frequency approaching $\pm \Delta$ [108]. In both cases, the spectral condensation enhances a tendency to chaos of the DS dynamics.

FIGURE 27.11 Contour-plots of $|\rho(z,t)|^2$. Left: Structurally chaotical complex of DSs (“soliton soup” or “soliton gas”). Right: Self-ordering of structural chaos (“condensation” of “soliton gas”). (Adapted from V. L. Kalashnikov, CSMIM, 1:51–59, 2011.)

FIGURE 27.12 The laminar-turbulent transitions in a fiber laser. Left: Clustering of continuous-wave light condensate (local power contour-plot, Adapted from E. G. Turitsyna, S. V. Smirnov, S. Sugavanam et al., Nat. Photon., 7:783–786, 2013.) Right: Self-similar clustering of DSs (local power contour-plot and its three-dimensional displays, Adapted from V. L. Kalashnikov, CSMIM, 1:29–37, 2014.)
27.3.2.2 Optical Rogue Waves

As has been demonstrated above, a local structural chaotization has granularity and turbulence as essential ingredients of phenomenon (compare with the intrasolar propagation analog in Reference 22 and a wave condensation in Reference 83). In this sense, both points of view (i.e., strong soliton interaction/colliding and intensity beatings induced by four-wave mixing) can be considered from a unified viewpoint.

The chaotically emerging DS intensity spikes, which are characterized by non-Gaussian statistics* of about ten fold (more than twofold by the standard definition [9]) excess of an eventual spike intensity over a stable (averaged) DS one, have been named as optical “rogue/freak-waves” or “extreme events” (e.g., see [149,182,220,228,262] and Figure 27.13).

This term belongs to rather rare but abnormally dangerous event in ocean navigation: an emergence of devastatingly huge wave (with \( \approx 30 \) m height!), which causes damage and even the loss of large ships (22 super carriers between 1969 and 1994) [137]. In a laser, such events cause an extremal spectral broadening, which mimics a CDS in average [149], but can also result in the damage of laser elements.

It should be noted that we did not still consider any noise or HOD (linear wave resonance) contribution to a DS dynamic. Nevertheless, the rogue DS exists under this condition. The linear wave resonance will be considered below, but an absence of noise contribution means that a rogue DS is deterministic by nature [31,149,183,235,262]; therefore, the term “noisy” DS in this context (see above) appears misleading. This statement does not mean that the stochastic processes do not contribute to rogue wave formations; there are examples, when the interplay between stochastic and multistable deterministic dynamics results rogue wave generation [67,195,219]. Noise can play the role of a “starter,” or the “switcher,” but the nature of rogue DS and, as a consequence, its statistical properties (i.e., extremal increase of the frequency of rare

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* That is, the large spikes appear much more often than they would occur according to Gaussian statistics: a “tail” of the distribution function is extremely stretched [9,203].

† In particular, the perturbation starts from a relatively narrow spectral component, then evolves into triangular spectrum shape (which is typical for a chaotic DS), and finishes into a “supercontinuum”-like spectrum with sharp edges [9].

‡ Nevertheless, rogue wave has an important feature of unpredictability: “it appears from nowhere and disappears without a trace” [9]. This type of rogue waves belongs to “intermittent-like rogue quasi-solitons that appear and disappear erratically” in terms of [83,182] and occur in the upper-left (“low-energy”) sector of the “master diagram” in Figure 27.5.
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events) are defined by nonlinear factors in a system. Simultaneously, a “deterministic nature” of optical rogue waves allows controlling their statistics and characteristics [65].

As was conjectured, the main mechanism of “rogue soliton” formation is a modulational instability [65,220], which can reveal itself in the formation of the so-called Akhmediev breathers in the framework of the nonlinear Schrödinger equation model [17,138,258]:

$$a(z,t) = \frac{(1 - 4m) \cosh(\eta z) + \sqrt{2m} \cos(\Omega t) + ib \sinh(\eta z)}{\sqrt{2m} \cos(\Omega t) - \cosh(\eta z)} \exp(iz),$$

(27.11)

where $\Omega$ is the dimensional modulation frequency, $m = 1/2(1 - \Omega^2/4)$ ($0 < m < 1/2$) determines the frequencies that experience gain, and $n = [8m(1 - 2m)]^{1/2}$ determines the instability growth.

The limiting case of the Akhmediev breather for $m \rightarrow 1/2$ corresponds to the so-called Peregrine (or “rational”) soliton (see Figure 27.14):

$$a(z,t) = \left[1 - \frac{4(1 + 2iz)}{1 + 4t^2 + 4z^2}\right] \exp(iz).$$

(27.12)

One can see that the Akhmediev breathers demonstrate a high degree of field localization, which is a characteristic feature of rogue solitons [83].

As was shown above, the chaoticity of DS is associated with the soliton granulation. In a fiber, such a granulation can be caused by HODs, which produce a resonant interaction of soliton with the linear waves (see below). As a result of granulation, the chaotic collisions between solitons occur (Figure 27.11, left). Such collisions provide an intensive energy exchange so that an extremal “supersoliton” can appear occasionally [78]. This mechanism can be considered as leading in the rogue DS formation. The important feature of both modulational instability and supersoliton scenarios of a rogue soliton formation is that they are very sensitive to a noise so that deterministic and noisy dynamics become strongly interwoven in a real-world system. In particular, such a noise transfer scenario can become leading when a stimulated Raman scattering (SRS) contributes to chaotic and rogue wave-like behavior of DSs (see below and [82]).

The practically important aspect of optical rogue waves is related to modern telecommunication systems. As has been shown [241], a rogue wave can be caused by a “purely linear statistical generation of huge amplitude waves.” It means that a “rogue wave” is not something thoroughly established but rather an emerging phenomenon within a broad context. Moreover, long-range (“nonlocal”) couplings induced, in particular, by SRS can substantially enhance a rogue wave-like behavior [46,82].

27.3.2.3 Nonlocal Structural Chaotization

In a real-world laser, there exist factors that can break a locality of dynamics. That is a current state of the field $a(z,t)$ is influenced by integral history of a system. These factors are, first of all, a dynamic gain (DG) and an SRS.

The first class of chaotic behavior is closely related with the DS frequency shift, which can be caused, for instance, by DG saturation, SRS, or line-width enhancement.\(^*\)

27.3.2.3.1 DG Saturation: Fine-Grain Aspect

In the framework of this aspect, a global (i.e., z-dependent) evolution of gain is not a decisive factor for DS dynamics. The DG saturation can be taken into account by a replacement of the gain term (i.e., the brace)

\(^*\) From a mathematical point of view, the class of underlying statistical models can be very broad and includes effectively linear ones [76].

\(^\dagger\) The frequency shift caused by HOD will be considered below in the framework of the model of resonance interaction with linear waves.

in Equation 27.2 by \( g(z, t) = g(z, 0) \left[ 1 - E_s^{-1} \int_0^t |a(z, t')|^2 \, dt' \right] \) [106,121], where \( g(z, 0) \) is the gain before a DS and can be expressed by the following map [104]:

\[
g(z + 1, 0) = g(z, 0) \exp \left( -\frac{\int_0^{T_{rep}} |a(z, t')|^2 \, dt'}{E_s} - \frac{T_{rep}}{U} - U \right) + \frac{g_m U}{U + \frac{T_{rep}}{U}} \left[ 1 - \exp \left( -\frac{T_{rep}}{U} - U \right) \right].
\]

Here, \( T_r \) is the gain relaxation time and \( U = \sigma_a T_{rep} I_p / h v_p \) is the dimensionless pump parameter \( I_p \) and \( v_p \) are the pump intensity and wavelength, respectively; \( \sigma_a \) is the pump absorption cross section. \( z \) and \( z + 1 \)-arguments mean the current and the next cavity round-trips, respectively. Equation 27.13 presupposes that the DS is well localized, i.e., the temporal window of its localization \( \ll T_{rep} \) and the soliton repetition period \( \approx T_{rep} \).

As was demonstrated in Reference 121, the pump growth causes a cascade of bifurcations so that regular oscillations of DS change into chaotic ones and vice versa. More careful study based on the numerical simulations in the framework of the model taking into account GDD and exact gain dynamics over the full resonator period [106] has demonstrated that the DG turns a semi-infinite parametric space of DS (Figures 27.5 and 27.6) into an isolated region (Figure 27.15) and, thereby, reduces the soliton energy scalability [55].

Figure 27.15 demonstrates an existence of two borders of DS stability: lower and upper. Lower border corresponds to a generation of satellite before a leading pulse due to preferential amplification of the pulse front caused by DG \( D \) in Figures 27.15 and 27.16. The born satellite interacts intensely with the DS and, as a result, their dynamics becomes chaotic. Further moving off the stability border (i.e., pump or GDD

\* This window defines the limits of integration in \( \int_0^{T_{rep}} |a(z, t')|^2 \, dt' \), where the lower limit (formally \( t = 0 \)) and the ultimate upper limit \( t = t_{max} \) correspond to the borders of this window.

FIGURE 27.16 DS temporal profiles (left) and spectra (right) corresponding to the points in Figure 27.15. (Adapted from V. L. Kalashnikov. Chirped-pulse oscillators: An impact of the dynamic gain saturation. Preprint arXiv:0807.1050v1 [physics.optics], available at http://lanLarxiv.org/abs/0807.1050, 2008.)

decrease) can cause either a continuum-wave (CW) generation with a DS “dissolution” or a regular two-soliton generation. For the last scenario, a cascade of further bifurcations with chaotic transits to a pulse multiplication is possible.

The upper stability border is characterized by the appearance of a CW perturbation with subsequent irregular interaction between this perturbation and DS (C in Figures 27.15 and 27.16). As a result, the DS becomes chaotic or even disintegrates. Note that the DS spectra are blue-shifted for both C- and D-regimes in Figure 27.16 because such a shift provides partial compensation (in the normal GDD range) of the time-advance and the preferential amplification of pulse front caused by DG saturation.

* The pulse multiplication is possible, also, without intermediate chaosization [129].
FIGURE 27.17 Spectral evolution of a Dissipative Raman soliton: numerical simulations of an all-normal-dispersion fiber laser with $\kappa = 0.1 \gamma$, $\zeta = 0.05 \gamma$, $\delta = 0.05$, a 40 nm spectral filter width, and $\beta = 0.2$ ps$^2$ (see Section 27.2). SRS parameters are: $T_1 = 12.2$ fs, $T_2 = 32$ fs, and $f_0 = 0.22$. This parametric point attributed to the master diagram of Figure 27.5 (left) corresponds to $C = 0.029$, $E = 142$ and lies on the boundary of a single Raman DS generation.

27.3.2.3.2 SRS: Raman DS

SRS can become a decisive factor in a high-energy DS dynamics. Its contribution can be taken into account by replacement of SPM terms in Equation 27.1 (i.e., $\gamma$- and $\chi$-terms) with [7]:

$$-i a(z, t) \int_0^\infty R(t') |a(z, t-t')|^2 dt', \quad (27.14)$$

where $R(t) = \gamma (1 - f_0) s(t) + \gamma f_0 h_R(t)$ and $h_R(t) = (T_1^2 + T_2^2) / T_1 T_2 \exp(-t/T_2) \sin(t/T_1)$. Here, $T_1$ is the inverse Raman resonance frequency, $T_2$ is the damping time of phonon vibrations, and $f_0$ represents the fractional contribution of the Raman response to a nonlinear polarization coefficient $\gamma$. Also, a noise source has to be added to Equation 27.1 in order to provide a spontaneous Raman scattering.

The numerical simulations [111,127] demonstrate that the SRS plays a substantial role in dynamics only for comparatively large energies and enhances a tendency to multipulsing. As a result, the stability boundary of single Raman DS adapted to the master diagram of Figure 27.5 (left) lies substantially lower, that is, the soliton stabilization requires a substantially larger GDD. The cause of destabilization is an enhancement of spectral loss due to SRS. Figure 27.17 demonstrated a single red-shifted Raman DS in the vicinity of the stability border. A distinguishing characteristic of such soliton is a strong perturbation (fragmentation) of its trailing edge caused by the growth of anti-Stokes spectral component (see Figure 27.18). As a result of this perturbation, the Raman DS demonstrates chaotic oscillations of peak power. This effect can be interpreted in the frameworks of concept of “incoherent soliton” [212] when there is enhancement of field perturbations via their long-range (in time and spectral domains) coupling [77]. The reverse side of such a coupling is suppression of wave turbulence in the presence of SRS (Figure 27.19).

27.3.2.3.3 Dynamic Loss Saturation and Linewidth Enhancement

A first law under brace in Equation 27.7 describes a perfectly saturable absorber. As a rule, that is a semiconductor structure with some finite response time $T_r$. Then, such a simple law is valid if the DS width $\gg T_r$. If not, the nonlinear loss and phase responses of such structure become nonlocal, i.e., it depends on $E > 20$ in terms of the master diagram presented in Figure 27.5, left.
energy flow (not instant power). Approximately, this effect can be taken into account by replacing Equation 27.7 with [123]:

$$- \mu \left[ 1 - (1 + i\epsilon) E_s^{-1} \int_0^t |a(z, t')|^2 dt' \right] \left[ 1 + \frac{1}{\Omega_i} \frac{\partial}{\partial t} \right]^{-1},$$

(27.15)

where $E_s$ is a loss saturation energy, $\Omega_i$ is an absorption linewidth, $\epsilon$ is a so-called linewidth enhancement (or Henry) factor [145], and a loss coefficient $\mu$ is not included in $\epsilon$. The differential operator has to be considered as an expansion in the Fourier domain.

As was demonstrated [122,123], such a system has a rich dynamics with multistability, auto-oscillations, and hysteresis. One can conjecture that the chaotic behavior also has to be present here. Moreover, the
linear term in Equation 27.15 induces HODs and, thereby, generates the dispersive waves, which interact resonantly with a DS and affect its dynamics (see below).

27.3.2.3.4 DG Saturation: Coarse-Grain Aspect

Above, the dynamical aspect of $z$-dependence of gain was neglected. However, such a dependence can be crucial in many cases. The powerful method for an analysis of this "coarse-grain" DG is the so-called nonlinear rate equations method, which reduce a dynamic phase-space to some low-dimensional map describing the evolution of DS parameters (intensity, energy, etc.) [104,126,155,173].

Figure 27.20 demonstrates the cascades of bifurcations with alternation of the chaos and regular dynamics in dependence on pump rate (left) or gain saturation energy (right).\footnotemark The left picture demonstrates that a region of DS existence can have an inhomogeneous structure with the slices of chaos and period multiplication. The right picture shows that a transit to multipulsing is accompanied by chaos and period multiplication, as well.

A numerical analysis taking into account a precise dynamics over a whole cavity period demonstrates that the multiple pulse generation can be chaotical, as well (Figure 27.21). The pulses can appear/disappear during $T_{rep}$ without a regular interpulse distance and relative amplitude. Strong interaction between the pulses in this case results from a DG saturation over a resonator period.\footnotemark

27.3.3 Chaotization Due to Resonant Interaction with Linear Waves

A soliton can be coupled with a linear wave only if the resonant condition is satisfied: a soliton wave number $q$ (see Equations 27.8 and 27.9) has to be equal to a linear wave one. There exists no "true soliton" in this case but a localized soliton with oscillating tail (or tails) can develop (a "quasi-soliton" in terms of References 257 and 259). In the framework of adiabatic theory, the resonant condition defines a CDS spectrum truncation in the absence of HODs: $q = \beta \Delta^2$.\footnotemark The contribution of HODs modifies this condition so that

\footnotetext[1]{That means that the time-scale of gain evolution is $\approx T_{rep}$ and is substantially larger than the soliton width. In this case, the DS influence is defined by its full energy (see Equation 27.2).}

\footnotetext[2]{Equation 27.2 demonstrates that both parameters are interrelated from the point of view of their contribution to dynamics.}

\footnotetext[3]{DG saturation plays a role of "long-range force" as opposed to "short-range" forces induced by DS overlapping.}

\footnotetext[4]{This condition is obtained in the framework of the adiabatic theory (see above) and explains the perturbation enhancement ("spectral condensation") along the DS spectrum edges (Figure 27.10 and [130]).}
q = \beta_{10}^2 + \sum_{n=1}^{N} \beta_n^{(n)} (\beta_n\text{-terms correspond to HOD contribution in Equation 27.1}). If such a resonance condition is satisfied and the corresponding resonance frequency \omega_0 lies inside the DS spectrum, the generation of dispersive wave begins, that is, the DS interacts resonantly with a continuum [108]. The spectral domain perturbation theory for this phenomena is presented in Reference 108. The numerical simulations demonstrate [117,128,221] that the resonance between a DS and a linear wave causes the strong chaotical perturbations of a soliton and, even, its fragmentation (Figure 27.22), which, nevertheless, preserves a localization of soliton complex (e.g., so-called “optical Newton’s cradles” [57]). This phenomenon is closely related to an “optical turbulence” [172,235] and is illustrated by the “wandering” phase trajectory in Figure 27.23 [222]. The dimension of embedding phase-space in Figure 27.23 lies within 2–4 and the attracting manifold has a toroidal shape. These facts are coherent with the concept of chaos induced by the nonlinearly entangled three oscillators [23,192] with the resonant frequencies \omega_0 (points R1, R2, and DW in Figure 27.24c). When the dispersive wave resonance frequency approaches the DS spectrum (point R2 in Figure 27.24c), the corresponding dispersive wave gets excited. Since the DW position corresponds to the positive group delay, the dispersive wave would propagate slightly faster than the DS and overlap with the pulse. The interference of dispersion wave and DS results in complete change of the whole spectrum and chaotic behavior (Figure 27.24). The spectrum and waveform of DS (Figure 27.24a, bottom and right-hand curves) become strongly modulated and change rapidly (Figure 27.24b). Yet the average spectrum, accumulated over 7000 round-trips, looks stable (Figure 27.24d).

Note that an HOD contribution also results from the limited gain/absorption/transmission spectral line widths (e.g., see Equation 27.15), which are inherent in any real-world optical setup. Therefore, this scenario of DS destabilization is an important factor limiting the soliton energy scalability.

27.4 Noises of Dissipative Solitons

The breakthroughs in optical solitonics shifted the attention from the physics of “power-envelope” to the physics of “spectral subtleties and phase” [50], from an “intensity” to a “field” [33], that is, from the classic...
27.4.1 Classical Noises

Basic theory of the noise characterization has been developed in Reference 243. The theory does not deal with the noise sources but concentrates on the spectral signatures of amplitude fluctuations and timing jitter (i.e., fluctuations of group velocity) of soliton. It was found that these fluctuations result in the different dependence of noise pedestal on the harmonic order in the radio-frequency (RF) spectrum of the pulse train: the amplitude fluctuations contribute to the frequency-independent term, whereas the timing jitter results in the quadratic dependence on RF frequency. Further analysis has revealed a more complicated picture [71]: there exists a linearity in frequency term due to the correlated jitter and energy fluctuations, while pulse width fluctuations contribute to the quadratic term, as well. Further generalization of the model, accounting for the cumulative character of timing fluctuations in a free-running passively mode-locked laser, connected the shape of the power spectrum of the pulse (DS) train with the type of noise (amplitude or timing jitter) and its correlative characteristics [66]. The picture turned out to be more complicated than it looked before. It has been demonstrated that the amplitude-phase coupling in a femtosecond laser [234] affects all aspects of stability, including the carrier-envelope-offset (CEO) stability [97,98] that, for instance, has a crucial significance for the optical frequency comb stabilization. Thus, an identification of noise sources and an establishment of correlation between them are required for an adequate model of the laser noises.

A physically clear picture of the frequency comb fluctuations induced by different physical mechanisms can be visualized by means of the so-called elastic tape model [30,178,230], where the comb frequencies are symbolized by the equidistant labels on an elastic tape, which is randomly stretched relative to some

*Optical frequency comb results from a periodical (≈T_{rep} or T_{rep}-harmonics) circulation of DSs in a laser. Thereby, a laser DS spectrum is “discretized” and can play a role of metrological standard if the CEO is stabilized. CEO is defined by a slip of the carrier wave, relatively, the DS envelope (such a slip defines a soliton wave number q) [50].
FIGURE 27.23  Phase portrait of the DS peak power evolution. The experimental set corresponds to 210 nJ of intracavity pulse energy of an Cr:ZnSe oscillator, the 2 μs lag corresponds to 290 cavity round-trips. (Adapted from E. Sorokin, N. Tolstik, V.L. Kalashnikov et al., Opt. Express, 21:29567–29577, 2013.)

FIGURE 27.24  Wigner function of chaotic DS (a) and corresponding round-trip phase and group-delay (c). Spectra (b) over the 7000 cavity round-trips and their accumulated spectrum (d). (Adapted from E. Sorokin, N. Tolstik, V.L. Kalashnikov et al., Opt. Express, 21:29567–29577, 2013.)
fixed point. Such a point can be localized in the vicinity of (i) zero frequency (e.g., cavity length fluctuations), (ii) carrier optical frequency (pump fluctuations or quantum noise), or (iii) higher-frequency domain (temperature fluctuations) [178, 190]. Knowledge of this pivot frequency (or frequencies) is crucial for an elaboration of successful comb stabilization strategy because the use of stabilizing technique fixing a “wrong” frequency point would even enhance the noise [167, 178, 190, 247]. Here, an integrated representation of the laser noises (both amplitude and phase) is required.

The outlook for such a representation has been outlined in the analytical theory by Paschotta [187, 190, 191]. Under the physically sound arguments, the theory allows explicitly connecting the source noise spectra with the temporal and phase noise spectra of a laser irrespective of details of a laser dynamics. The direct generalization of this theory [115] has predicted the timing jitter suppression in a thin-disk multimicrojoule Yb:YAG oscillator operating in the normal GDD regime (Figure 27.25, left).

The situation becomes more intricate for fiber lasers where the stimulated Raman, Brillouin, and Rayleigh scattering as well as the stochastic birefringence can affect the stochastic evolution substantially [96, 237, 246]. Difference in the time and spatial scales characterizing linear, nonlinear, and stochastic evolution begins playing a crucial role in dynamics and allows applying the multiscale and kinetic theory methods for an analysis of optical phenomena [73, 175, 180]. As a result of the joint action of multiscale factors, the quite new phenomena can appear: a “stochastic resonance,” when a noise enhances a system response to the external periodic perturbation [72, 84, 156], a “coherence resonance,” when a noise creates some coherent-like states in a system [156], or a “stochastic antiresonance,” when the fluctuations intensify resonantly within some diapason of system parameters where a switching between the statistical scenarios (e.g., persistent and antipersistent) occurs. Wealth of the phenomena inherent to the incoherent nonlinear wave propagation stimulates developing a new branch of nonlinear optics, the so-called “stochastic nonlinear optics” [76], and a concept of “incoherent soliton” [212].

Back on topic of the DS noises, the perturbation theory of the frequency comb generated by a solitonic laser has been developed in Reference 179. The theory based on the MM (see above) has allowed connecting the offset and spacing fluctuations of the frequency comb with the factors governing the laser dynamics. In particular, the transfer functions have been derived, which relate the comb fluctuations with the fluctuations of cavity length and pump power. However, the success of this theory should not be overestimated because data about independence of the noise and the mode-locking regime are controversial (e.g., see [35, 189]). Therefore, there is a need for an extensive numerical model that accounts for all factors

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involved in the laser dynamics and imposes no a priori restrictions on the complex field profile $a(z, t)$. An implementation of such modeling has been presented in References 115, 187, 189, and 190. In particular, the numerical simulations of a high-energy thin-disk Yb:YAG oscillator confirmed the timing jitter suppression in the normal GDD regime predicted by the analytical theory (Figure 27.25, left). The details of numerical approach to the noise modeling are more appropriately considered in the following subsection examining the quantum laser noises.

27.4.2 Quantum DS Noises

Unprecedented precision and stability levels required from ultrafast lasers generating DSs compel reducing the noise level down to quantum limit. Therefore, the study of such a limit is urgent for both applications and fundamental physics of DSs.

27.4.2.1 Soliton Quantization

The well-established basis for a quantum theory of noises has been provided by the quantum noise theory of a linear (or linearized) amplifier [47,87,90–92]. It is shown in this theory that the "phase-preserving" character of amplification requires quantum noise for conservation of the field commutators. This means that contribution from additional degrees of freedom is required by the "phase-preservation": these include spontaneous emission, intracavity loss (with the corresponding "baths"), and output coupling of an open oscillator (i.e., contribution from the "input" vacuum fluctuations). The statistical properties of the observables connected with the noise operators can be obtained from the commutator conservation law.

The resulting dynamical equations have a form of Langevin operator equations with the quantum noise sources [29,34,75,81,91,99,254]. An implementation of the fluctuation-dissipation theorem and the field uncertainty for an evaluation of the quantum Langevin equations has been reviewed in Reference 99. Thereupon, the operator Langevin equations can be reduced to the stochastic differential ones, where the field and the gain are driven by the integrated (in the Ito or Stratonovich forms) stochastic forces with the diffusion coefficients defined by correlation functions of the noise sources [74,99,153]. There exist different methods for obtaining these coefficients (for an overview, see, for instance, [99]). In particular, the quantum Langevin approach reproduces the Schawlow–Townes limit on the CW laser linewidth [208] (which results also from the commutation relations imposed on the noise sources [252]) as well as provides the powerful methods for laser noise interpretation [85] and its suppression [34,100].

To extend the quantum noise theory to the mode-locked lasers, one would additionally have to consider all nonlinear effects, which come along with the DS propagation. This requires quantization of the dynamically nontrivial solitonic sector of the evolution equations. This approach can be viewed as a part of a more general quantum-field theory [68]. The starting point for an optical soliton quantization is quantization of the nonlinear Schrödinger equation [61,91]. Since the classical nonlinear Schrödinger equation is exactly integrable by the inverse scattering method, it can be exactly quantized, as well [135,152,232]. Formally, the problem is equivalent to the N-body model with a point interaction between bosons [151,231,256].

It is also possible to solve the problem of quantization by linearization of the Heisenberg quantum form of the nonlinear Schrödinger equation around the exact classical soliton solution [91,93,134,150]. This approach (quantum soliton perturbation theory) has allowed further generalization, accounting for the dissipation [197] and the gain [86] contributions to the Langevin noise terms as well as description of a repeater system model of an oscillator. The perturbation theory has been extended to the cubic NCGL equation governing the dynamics of a passively mode-locked laser [88,89] (an actively mode-locked laser is describable in the framework of the linear Heisenberg–Langevin model [199]). The perturbations of pulse energy, phase, timing, and frequency have been expressed from the characteristics of noise sources: both technical (gain, resonator length, and refractive index fluctuations) and quantum (spontaneous emission). This approach has allowed derivation of the noise spectra and the correlation functions of actively and passively mode-locked lasers.
The noise transfer matrix obtained from the semiclassical perturbation theory allows precise interpretation and its matrix elements are measurable [168,241,245]. As a result, the expressions for noise spectrum and the frequency comb linewidths can be obtained semiphenomenologically for a given laser configuration. The theory limits itself to the case of a dispersion-managed soliton, operating in the anomalous dispersion regime and in the absence of higher-order dispersions and nonlinearities. While a generalization of this approach to the normal dispersion regime and more complicated dispersion and nonlinearity functions seems to be principally possible, this has not been performed yet. Nevertheless, an important feature of this approach is the demonstrated principal measurability of the transfer matrix elements, which provides an independent consistency check route for the model, analytical or numerical.

An important contribution to the theory of noises of mode-locked oscillators is the semiclassical (“phase-preserving”) analytical models of References 168, 187, and 191, which reproduce the results of [88,89,199] based on the soliton perturbation theory. The key advantage of the semiclassical model is that it does not restrict itself to the soliton regime and, thereby, its area of applications is broader. In particular, it is applicable for the description of an oscillator operating in the normal-dispersion regime (fiber-based [189] or solid-state [115]), where an ordinary soliton would not exist. The extensive study of CDS formed in the all-normal-dispersion regime has demonstrated that the timing quantum noise can be strongly reduced (see Figure 27.25, right) [115].

The quantization schemes for a CDS have not been developed systematically to date. The main problem is that the relevant nonlinear evolution equations are not integrable. A possible breakthrough is expected due to further development of the approximated integration techniques [110] and the perturbation methods in the spectral domain [108].

Despite the significant progress achieved on the basis of the soliton-perturbation and semiclassical analytical theories of quantum noises in mode-locked oscillators, these theories are insufficient because they neglect the complicated dynamics of the real-world (especially fiber-based) oscillators and the mutual correlation of various noise mechanisms. Therefore, extensive numerical simulations of the mode-locked oscillators with the inherent noise sources are required.

27.4.2.2 Semiclassical Langevin Methods: Numerical Approach

The numerical model based on the Langevin semiclassical method allows including the spontaneous emission of noise source directly into the classical equations for laser field, population difference, and complex polarization [70,119,131]. The noise results from fluctuation of the dipole moment in an active medium. As an example of results, one can note the demonstration of the “seed-points” inside a mode-locking region with maximum suppression of the noise-induced amplitude fluctuations [53] (“coherence resonance” contributing to a system self-ordering [156]). It was also found that the mode-locking threshold depends on the correlation properties of the noise [118]. The Langevin method also allows easy accounting for technical noises.

The numerical Langevin method based on statistics gathering has recently seen a renaissance [115, 186,187,189] made possible by ever-growing computing resources. In particular, it is becoming feasible to model long pulse trains, which is a necessary condition for the reconstruction of a low-frequency edge of the noise spectrum. The numerical approach has allowed accounting for loss saturation in a semiconductor saturable absorber, the nonlinear dispersion, as well as relatively low-frequency gain relaxation dynamics and related gain dispersion. As a result, the noise coupling mechanisms have been observed, which are not present in the simple analytical models. In particular, these coupling mechanisms cause the excess phase noise resulting in the line broadening above the Schawlow–Townes limit, as well as the linewidth variation across the frequency comb [190] predicted on the basis of the perturbation theory [244]. The different types of the fiber lasers have also been briefly surveyed [189]. It was found that the stretched-pulse modelocking does not introduce strong excess noise. In the wavebreaking-free fiber lasers (i.e., lasers operating in the all-normal-dispersion regime), the center frequency noise can be effectively coupled to the timing noise that, in combination with the long pulse width, causes a strong excess enhancement of the timing and
carrier-envelope offset noises [189,198]. This contradicts the case of comparatively narrow-band thin-disk solid-state laser [115] and the data of References 35 and 209.

An important advantage of the numerical approach is that it allows precise modeling of the intracavity field evolution in the resonators with discrete or strongly inhomogeneous elements. Such evolution is most pronounced in nonsoliton fiber lasers: DS, similariton, and similariton-soliton types [181,201,250]. These types of fiber lasers are of special interest because they provide the energy scalability of DS.

Some pilot results for the lasers with lumped elements can be obtained analytically on the basis of the VA or the MM (see above). It has been shown, that the dynamics of a system, i.e., the evolution of pulse parameters during a cavity round-trip, can cause significant noise reduction because of noise decoupling [162,177]. On the other hand, the fixed frequency points can shift, as well [178].

27.4.2.3 Quasi-Probability Methods
These methods are based on the quantum phase-space formulation [4-6] of nonlinear quantum optical pulse propagation in the form of the positive-P or Wigner distributions [37,58,60,139,217]. The quantum dynamics of radiation, described in the form of the nonlinear Liouville equation, can be reduced to a set of stochastic differential equations for the classical functions (not operators) describing the P or Wigner quasi-probability operator representations [36,49,59]. In spite of some formal distinctions, this method is equivalent to the linearized stochastic approach above in the soliton quantization scheme [69].

The Wigner representation can be considered as a rigorous substantiation of the semiclassical Langevin methods, described in the previous subsection. However, one has to keep in mind that the Wigner representation (i) is approximated, though practically perfect for large photon numbers, (ii) corresponds to symmetrically (not normally) ordered operator products and has to be corrected to provide a direct interpretation. In particular, the vacuum noise terms have to be included additionally: "shot-noise" as an initial condition, and both gain and loss as complex additive noises [36,49,59]. The advantage of this representation is its reduced dimensionality in comparison with the positive P-representation (e.g., see [165]). At the same time, the latter representation is exact, has no initial and absorbing reservoir vacuum noise terms, and corresponds to the experimentally measurable field averages. Therefore, despite its double dimension, using the P-representation has a big potential.

As an alternative approach to the quasi-probability methods, the modern computational resources allow simulating the dissipative Liouville equation [229] without reduction of quasi-probability representations to a set of stochastic equations. The main advantage of this method is that there is no need to gather the statistics because all required information is contained in the evolving quasi-probability distribution. One may believe that this technique becomes especially effective for the distributed resonators with long-living upper-laser level media, i.e., fiber lasers, where statistics gathering becomes prohibitively expensive computationally. An additional advantage of a direct evaluation of quasi-probability distributions is the possibility to include the measurement process into analysis using the Wigner (or another distribution) function of a measuring device [154].

27.5 Conclusion
The concept of dissipative soliton has become well established over the last two decades. Dissipative solitons, which are localized structures preserving a self-identity during a long evolution, describe an enormously broad range of phenomena ranging from physics to biology, and geophysics to social sciences. An existence far from the equilibrium state of a system results in a substantial nontriviality of dissipative soliton dynamics, which can reveal itself in chaotic, multistable, and extremely noise-sensitive behavior. Thereby, possessing the "energy" (or "mass") scalability (i.e., the definitely "macroscopic" property), a dissipative soliton can be "quantum-sensitive." A dissipative soliton is a "mesoscopic" object and the study of its properties is of fundamental interest.
The impressive progress of modern ultrashort laser pulse technologies provides us with a unique test-bed for dissipative soliton exploring. The studies of such optical dissipative solitons advance with sevenleague strides. In this short review, only some aspects of optical dissipative soliton dynamics have been considered. The main approaches to dissipative soliton theory have been exposed and the preliminary classification of scenarios of an optical dissipative soliton chaotization has been proposed. Also, the main trends in the studies of noise influence (including quantum one) on an optical dissipative soliton have been reviewed. It should be recalled that the unified theory of optical dissipative solitons remains under way and one may expect new impressive advantages in this field in the near future.

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