Dissipative Solitons: The Structural Chaos
And The Chaos Of Destruction

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Abstract. Dissipative soliton, that is a localized and self-preserving structure, develops as a result of two types of balances: self-phase modulation vs. dispersion and dissipation vs. nonlinear gain. The contribution of dissipative, i.e. environmental, effects causes the complex “far from equilibrium” dynamics of a soliton: it can develop in a localized structure, which behaves chaotically. In this work, the chaotic laser solitons are considered in the framework of the generalized complex nonlinear Ginzburg-Landau model. For the first time to our knowledge, the model of a femtosecond pulse laser taking into account the dynamic gain saturation covering a whole resonator period is analyzed. Two main scenarios of chaotization are revealed: i) multipusing with both short- and long-range forces between the solitons, and ii) noiselike pulse generation resulting from a parametrical interaction of the dissipative soliton with the linear dispersive waves. The noiselike pulse is characterized by an extremely fine temporal and spectral structure, which is similar to that of optical supercontinuum.

Keywords: Dissipative soliton, Complex nonlinear Ginzburg-Landau equation, Chaotic soliton dynamics.

1 Introduction

The nonlinear complex Ginzburg-Landau equation (NCGLE) has a lot of applications in quantum optics, modeling of Bose-Einstein condensation, condensate-matter physics, study of non-equilibrium phenomena, and nonlinear dynamics, quantum mechanics of self-organizing dissipative systems, and quantum field theory [1]. In particular, this equation being a generalized form of the so-called master equation provides an adequate description of pulses generated by a mode-locked laser [2]. Such pulses can be treated as the dissipative solitons (DSs), that are the localized solutions of the NCGLE [3]. It was found, that the DS can demonstrate a highly non-trivial dynamics including formation of multi-soliton complexes [4], soliton explosions [5], noise-like solitons [6], etc. The resulting structures can be very complicated and consist of strongly or weakly interacting solitons (so-called soliton molecules and gas) [7] as well as the short-range noise-like oscillations inside a larger wave-packet [8]. The nonlinear dynamics of these structures can cause both regular and chaotic-like behavior.

In this article, the different scenarios of the soliton structural chaos will be considered for the chirped DSs formed in the all-normal group-delay dis-
persion region [9]. The first scenario is an appearance of the chaotic fine graining of DS. For such a structure, the mechanism of formation is identified with the parametric instability caused by the resonant interaction of DS with the continuum. The second scenario is formation of the multi-soliton complexes governed by both short-range forces (due to solitons overlapping) and long-range forces (due to gain dynamics). The underlying mechanism of formation is the continuum amplification, which results in the soliton production or/and the dynamical coexistence of DSs with the continuum.

2 Chirped dissipative solitons of the NCGLE

Formally, the NCGLE consists of the nondissipative (hamiltonian) and dissipative parts. The nondissipative part can be obtained from variation of the Lagrangian [10]:

\[
\mathcal{L} = \frac{i}{2} \left[ A^* (x, t) \frac{\partial A(x, t)}{\partial t} - A(x, t) \frac{\partial A^*(x, t)}{\partial t} \right] + \\
+ \frac{\beta}{2} \frac{\partial A(x, t)}{\partial t} \frac{\partial A^*(x, t)}{\partial t} - \frac{\gamma}{2} |A(x, t)|^2,
\]

where \( A(x, t) \) is the field envelope depending on the propagation distance \( x \) and the “transverse” coordinate \( t \) (that is the local time in our case), \( \beta \) is the group-delay dispersion (GDD) coefficient (negative for the so-called normal dispersion case), and \( \gamma \) is the self-phase modulation (SPM) coefficient [11].

The dissipative part is described by the driving force:

\[
\mathcal{Q} = -i\Gamma A(x, t) + i \frac{\rho}{1 + \sigma \int_{-\infty}^{\infty} |A|^2 \, dt'} \left[ A(x, t) + \tau \frac{\partial^2 A(x, t)}{\partial t^2} \right] + \\
+ i\kappa \left( |A(x, t)|^2 - \zeta |A(x, t)|^4 \right) A(x, t),
\]

where \( \Gamma \) is the net-dissipation (loss) coefficient, \( \rho \) is the saturable gain \( \sigma \) is the inverse gain saturation energy if the energy \( E \) is defined as \( E \equiv \int_{-\infty}^{\infty} |A|^2 \, dt' \), \( \tau \) is the parameter of spectral dissipation (so-called squared inverse gainband width), and \( \kappa \) is the parameter of self-amplitude modulation (SAM). The SAM is assumed to be saturable with the corresponding parameter \( \zeta \).

Then, the desired CNGLE can be written as

\[
i \frac{\partial A(x, t)}{\partial x} - \frac{\beta}{2} \frac{\partial^2 A(x, t)}{\partial t^2} - \Gamma A(x, t) - \gamma |A(x, t)|^2 A(x, t) = \\
= -i\Gamma A(x, t) + i \frac{\rho}{1 + \sigma \int_{-\infty}^{\infty} |A|^2 \, dt'} \left[ A(x, t) + \tau \frac{\partial^2 A(x, t)}{\partial t^2} \right] + \\
+ i\kappa \left( |A(x, t)|^2 - \zeta |A(x, t)|^4 \right) A(x, t).
\]
Eq. (3) is not integrable and only sole exact soliton-like solution is known for it [11,12]. Nevertheless, the so-called variational method [10] allows exploring the solitonic sector of (3). The force-driven Lagrange-Euler equations

\[ \frac{\partial}{\partial t} \int_{-\infty}^{\infty} \mathcal{L} \, dt - \frac{\partial}{\partial x} \frac{\partial}{\partial f} \int_{-\infty}^{\infty} \mathcal{L} \, dt = 2 \Re \int_{-\infty}^{\infty} \frac{\partial \mathcal{A}}{\partial f} \, dt \]  

(4)

allow obtaining a set of the ordinary first-order differential equations for a set \( f \) of the soliton parameters if one assumes the soliton shape in the form of some trial function \( \mathcal{A}(x, t) \approx \mathcal{F}(t, f) \). One may chose \[ \mathcal{F} = a(x) \text{sech} \left( \frac{t}{T(x)} \right) \exp \left[ i \left( \phi(x) + \psi(x) \ln \left( \text{sech} \left( \frac{t}{T(x)} \right) \right) \right) \right], \]  

(5)

with \( f = \{a(x), T(x), \phi(x), \psi(x)\} \) describing amplitude, width, phase, and chirp of the soliton, respectively.

Substitution of (5) into (4) results in four equations for the soliton parameters. These equations are completely solvable for a steady-state propagation (i.e. when \( \partial_x a = \partial_x T = \partial_x \psi = 0, \partial_x \phi \neq 0 \)). The analysis demonstrates that the solitonic sector can be completely characterized by two-dimensional master diagram, that is the DS is two-parametrical and the corresponding dimensionless parameters are \( c \equiv \tau \gamma / |\beta| \kappa \) and the “energy” \( \mathcal{E} \equiv E b \sqrt{\kappa \zeta / \tau} \) (here \( b \equiv \gamma / \kappa \)).

The master diagram is shown in Fig. 1. The curves correspond to the stability threshold defined as \( \Gamma - \rho / \left( 1 + \sigma \int_{-\infty}^{\infty} |\mathcal{A}|^2 \, dt \right) = 0 \). Positivity of this value provides the vacuum stability. As will be shown, the vacuum destabilization is main source of the soliton instability causing, in particular, the chaotic dynamics. The master diagram in Fig. 1 reveals a very simple asymptotic for the maximum energy of DS: \( E \approx 17 |\beta| / \sqrt{\kappa \zeta \tau} \). The continuum rises above this energy.

### 3 Resonant excitation of continuum

The stability threshold shown in Fig. 1 corresponds to an unperturbed DS of (3). The physically meaningful perturbation results from a higher-order dispersion correction to the Lagrangian: \( \mathcal{L} = \mathcal{L}_0 + i \delta \frac{\partial^2 A}{\partial \tau^2} \frac{\partial A}{\partial \tau} \), where \( \mathcal{L}_0 \) is the unperturbed Lagrangian (1) and \( \delta \) is the third-order dispersion (TOD) parameter.

The DS of unperturbed Eq. (3) does not interact with the continuum and the collapse-like instability is suppressed by \( \zeta > 0 \). Nevertheless, the DS peak power on the asymptotic stability threshold of Fig. 1 is \( \approx 1.1/\zeta > 1/\zeta \) that, in accordance with [8] has to result in the chaotic behavior. However, such a chaotic layer in the vicinity of stability threshold has not been revealed by our numerical simulations. A possible explanation is that the solitonic sector (5)
is not isolated and there exists another stable solitonic sector corresponding to the DS with the so-called “finger-like” spectra [9,14]. Such a spectrum has a main part of the power in the vicinity of the spectrum center. As a result, a spectral loss decreases that corresponds to energy growth close to the boundary of the DS stability. In turn, a concentration of power close to the spectrum center corresponds to a similar power concentration around a pulse peak in the time domain, that allows the stable DSs with the peak powers $>1/\zeta$ and, thus, a new solitonic sector appears. Such a sector corresponds to

$$\mathcal{F} = \frac{a(x)}{\sqrt{\theta(x) + \cosh \left( \frac{t}{T(x)} \right)}} \exp \left[ i \left( \phi(x) + \psi(x) \ln \left( \theta(x) + \cosh \left( \frac{t}{T(x)} \right) \right) \right) \right]$$

with $\theta(x) > 1$ and requires a further exploration.

Nevertheless for any solitonic sector, $\delta \neq 0$ can result in an appearance of interaction with the continuum at some frequency [15]. Hence, the stability threshold becomes lower than that shown in Fig. 1. As the resonance occurs in the spectral domain, an exploration of the DS spectrum is most informative in this case. The numerical results corresponding to a mode-locked Cr:ZnSe oscillator [16] are shown in Fig. 2. Non-zero $\delta$ can be treated as a frequency-dependence of net-GDD (dashed curves). As a result of such dependence, the zero GDD shifts towards the DS spectrum with the growing $|\delta|$ (black solid curve and black dashed line correspond to $\delta = 0$). The vertex of truncated DS spectrum (solid curves) traces the GDD line (dashed lines) logarithmically. When the zero GDD (see orange dashed line) reaches the DS spectrum, the resonant generation of continuum (dispersive wave) begins (longer wavelength
side of the spectrum demonstrates such a wave; orange and violet curves in Fig. 2).

![Graph](image)

**Fig. 2.** Spectra of the chirped DSs (solid curves) corresponding to the different net-GDD (dashed curves). The GDD slope depends on the TOD value.

As a result of resonant interaction with the dispersive wave, when the zero GDD approaches the central wavelength of an unperturbed DS spectrum (≈2.5 µm in our case), the irregular beatings of the DS peak power develop (violet curve in Fig. 3, left). In the time domain, the resonant interaction forms a fine (femtosecond) structure in the vicinity of pulse peak (Fig. 3, right). Such a structure enhances with the shift of zero GDD towards the central wavelength of an unperturbed DS spectrum and spreads to a whole spectrum. As a result, the DS envelope becomes to be strongly distorted (gray curve Fig. 3, right) and disintegrates.

![Graph](image)

**Fig. 3.** Peak power evolution (left) and DS envelopes (right) for the GDD curves of Fig. 2.
4 Nonresonant excitation of continuum

Another correction to (3) is defined by the energy-dependent gain/loss terms in (2). As a result, a large-scale solitonic (multi-soliton) structure appears (Fig. 4) and, the picosecond satellites appear both nearby (few picoseconds) the main pulse and far (nanoseconds) from it. Strong interaction between the pulses with the contribution from a gain dynamics results in a chaotic behavior. For the chirped DS, the dynamic loss/gain saturation causes a parametric resonance, as well. Hence, the DS becomes finely structured [17].

![Fig. 4. Multiple DS evolution in the presence of the dynamic gain saturation.](image)

5 Conclusion

Unlike a classical soliton, a chirped DS posses a nontrivial internal structure. As a result, the dynamics of such DS can be very complicated. In particular, a chaotic interaction with an excited vacuum (continuum) develops. Such an interaction can be nonresonant (as it takes a place for the Schrödinger soliton) and resonant. The last results in the chaotic behavior with a strict localization of the DS spectrum and power envelope even for a “far from equilibrium” regime. The strong localization of a chaotic structure results in the chirped DS, which remains to be traceable in an even chaotic regime. Such a traceability promises a lot of applications in the spectroscopy, for instance.

Acknowledgements

This work was supported by the Austrian Science Foundation (FWF project P20293).
References