The unified theory of chirped-pulse oscillators

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ABSTRACT

A completely analytical theory of chirped-pulse oscillators is presented. The theory is based on an approximate integration of the generalized nonlinear complex Ginzburg-Landau equation. The obtained parametric space of a chirped-pulse oscillator allows easy tracing the characteristics of both solid-state and fiber oscillators operating in the positive dispersion regime.

Keywords: chirped-pulse oscillator, positive-dispersion regime, nonlinear Ginzburg-Landau equation, dissipative soliton

1. INTRODUCTION

In the last decade, femtosecond pulse technology has evolved rapidly and allowed achieving a few-optical-cycle pulse generation directly from an oscillator.\(^1\) Applications of such pulses range from medicine and micromachining to fundamental physics of light-matter interaction at unprecedented intensity level and time scale.\(^2\)\(^-\)\(^5\) High-energy laser oscillators nowadays have reached the intensity level of the order of \(10^{14}\) \(\text{W/cm}^2\), which allows high-intensity experiments such as direct gas ionization.\(^6\) To achieve these regimes, short- and over-microjoule pulse energies are required. Such energy frontiers have become achievable due to the chirped pulse amplification outside an oscillator.\(^2\) However, the amplifier technology is i) complex, ii) expensive, iii) accessible pulse repetition rates lie within the kHz range, and iv) noise amplification is unavoidable.

It is desirable to find a road to the direct over-microjoule femtosecond pulse generation at the MHz pulse repetition rates without an external amplification. To date, a promising approach has been proposed. It is based on a considerable decrease of the oscillator repetition rate.\(^7\)\(^,\)\(^8\) The catch is that a long-cavity oscillator suffers from strong instabilities caused by enhanced nonlinear effects owing to increase of the pulse peak power \(P(0)\). The leverage is to stretch a pulse and, thereby, to decrease its peak power below the instability threshold. Recently, a critical milestones, demonstrating the feasibility of this approach, has been achieved for the Ti:sapphire oscillators operating both in the negative- (NDR)\(^9\)\(^,\)\(^10\) and positive-dispersion regimes (PDR),\(^11\)\(^,\)\(^12\) the near-infrared Yb:YAG thin-disk oscillators operating in the NDR,\(^13\)\(^-\)\(^15\) and the fiber oscillators operating in the all-normal dispersion (ANDi) regime (that is the PDR by definition).\(^16\)

The fundamental difference between the NDR and the PDR is that, in the first one, the Schrödinger soliton develops.\(^17\) Since the soliton peak power \(P(0)\) has to be lower than some threshold value \(P_{th}\) in order to avoid the soliton destabilization, the maximum reachable energy can be estimated as \(E = 2P_{th}T\) (\(T\) is the soliton width). That is the energy scaling requires the pulse stretching. However, the latter results from substantial growth of the group-delay dispersion (GDD) (quadratically with energy\(^18\)). As a result, the energy scaling requires a huge negative GDD, the obtained chirp-free soliton has a large width, and it is not compressible linearly.

In the PDR,\(^19\)\(^,\)\(^20\) the pulse is stretched and its peak power reduced due to large chirp.\(^21\)\(^,\)\(^22\) The chirp compensates the spectrum narrowing with energy and the pulse becomes to be compressible linearly down to \(T \approx 2/\Delta\), where \(\Delta\) is the spectrum half-width. The issue is that the chirped solitary pulse (CSP) is a dissipative soliton, that is it develops in a dissipative nonlinear system and, as a result, there is no a uniform description of its properties and dynamics, because the underlying nonlinear equation (so-called, nonlinear complex Ginzburg-Landau equation, CGLE) is not integrable.\(^23\)\(^,\)\(^24\)

In this work, I propose the approximate method of integration of the generalized nonlinear CGLE and demonstrate that the CSP is its solitary pulse solution with reduced dimension (2 or 3) of the parametric space.

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As a result, the CSP characteristics become easily traceable on the two-dimensional diagram (“master diagram”). Comparison of the PDR parameters demonstrates that the CSPs formed in the ANDi fiber oscillator and in the CPO: i) lie within the distinct sectors of the unified master diagram, ii) belong to the distinct branches of solution, and iii) vary with parameters in different ways. Comparison of the models based on the different versions of the master equation is carried out. The phenomenon of concave spectrum is attributed to the quintic self-phase modulation.

2. DISSIPATIVE SOLITON OF NONLINEAR CGLE

The nonlinear CGLE is the generalized form of the master mode-locking equation\textsuperscript{17,23,25} and provides an adequate description of mode-locked oscillators (both fiber and solid-state). Its soliton-like solutions (or dissipative solitons) pattern the laser pulses. Such an approach is well-grounded if i) \( T \gg 2\pi/\omega_0 \) (\( \omega_0 \) is the carrier frequency of laser field) and ii) relative variation of laser field during one cavity round-trip is small.

Let \( u(z,t) \) be a slowly-varying field amplitude, \( z \) be a propagation coordinate normalized to the cavity period, \( t \) be a local time. The generalized CGLE is

\[
 u_z = -\sigma u + (\alpha + i\beta) u_{tt} - iu \left( \gamma |u|^2 + \chi |u|^4 \right) + uf \left( |u|^2 \right),
\]

where \( |u|^2 \) is the instant power, \( \sigma \) is the saturated net-loss, and \( \alpha \) is the squared inverse spectral bandwidth of oscillator (as a rule, it is defined by gain bandwidth). Parameter \( \beta \) is the net-GDD coefficient; \( \gamma \) is the self-phase modulation (SPM) coefficient, and \( \chi \) describes a high-order correction to it (i.e. the quintic SPM). Function \( f \left( |u|^2 \right) \) models the self-amplitude modulation (SAM) in an oscillator and its form depends on the mode-locking mechanism.

The CSP develops in the PDR under combined action of two mechanisms: the pure phase and dissipative ones. The first one results from a balance of phase contributions from the pulse envelope \( \beta u_{tt} \) and the time-dependent phase \( -\beta u (\phi_t)^2 \). Such a balance is provided by some value of pulse chirp. However, a sole phase balance is not sufficient as the pulse spreads. The spreading can be compensated by spectral filtering. Since the chirp causes the frequency deviation at pulse front and tale, the filter cuts off the higher- and lower-frequency wings of pulse and, thereby, shortens it.\textsuperscript{20,21}

The partial exact CSP solution of Eq. (1) is known for \( f \left( |u|^2 \right) = \kappa \left( 1 - \zeta |u|^2 \right) |u|^2 \) (cubic-quintic nonlinear CGLE, for overview see\textsuperscript{23,24}). In this work, the approximate method of integration of Eq. (1) in a general form will be proposed. The underlying approximations are

**Conjecture 2.1.** \( T \gg \sqrt{\beta} \), that is the adiabatic approximation;

**Conjecture 2.2.** \( \beta \gg \alpha \), that is the GDD prevails over the spectral dissipation.

The first conjecture is valid for both solid-state and fiber oscillators operating in the PDR because the pulse is strongly stretched in the regime under consideration. The second conjecture is valid for both broadband solid-state (i.e. Ti:Sapphire\textsuperscript{22} and Cr:YAG\textsuperscript{26}) and fiber oscillators.\textsuperscript{27} Thin-disk oscillators based on the narrowband active media\textsuperscript{13} can approach the limit of \( \beta \approx \alpha \) and this issue will be addressed in Subsection 2.4.

2.1. Cubic nonlinear CGLE

The simplest version of (1) corresponds to \( \chi = 0 \) and the SAM function\textsuperscript{21}

\[
 f \left( |u|^2 \right) = \kappa |u|^2.
\]

Such a function approximates nonlinearity of low-energy solid-state and fiber oscillators. The \( \kappa \)-parameter describes a nonlinear gain due to loss saturation. Eqs. (1,2) lead to a dissipative generalization of the nonlinear Schrödinger equation.

Let’s make the traveling wave reduction of Eqs. (1,2) by means of ansatz
\[ u(z,t) = \sqrt{P(t)} \exp[i\phi(t) - iqz], \quad (3) \]

where \( P \) is the \( z \)-independent instant power, \( \phi(t) \) is the time-dependent phase, and \( q \) is the phase due to slip of the carrier phase with respect to the envelope. In contrast to the tradition approach, we do not impose hereafter any restriction on the time-dependence of phase. Substitution of (3) in (1) supplemented with (2) as well as taking into account the approximations under consideration and the obvious restrictions \( P > 0 \) and \( \phi_{tt} < 0 \) lead to

\[
\gamma P(t) = \beta \Delta^2 \left( 1 - \tanh^2 \left[ t \kappa (1 + c) \Delta/3\gamma \right] \right),
\]

\[
\alpha \Delta^2 = \frac{3\sigma c}{2 - c}, \quad \gamma P(0) = \beta \Delta^2, \quad (4)
\]

where \( \Delta^2 \equiv q/\beta \), \( P(0) \) is the CSP peak power, and the control parameter \( c \equiv \alpha \gamma/\beta \kappa \). One can see, that, on conditions that the appropriate normalizations are used, the CSP is two-parametric and depends on only \( \sigma \) and \( c \).

Since a pulse is strongly chirped in the PDR, one may treat \( \phi \) as a rapidly varying function and apply the method of stationary phase to the Fourier image of \( u \). As a result, the spectral power is

\[
p(\omega) \equiv |e(\omega)|^2 \simeq \frac{6\pi \beta}{(1 + c) \kappa} H(\Delta^2 - \omega^2), \quad (5)
\]

where \( e(\omega) \equiv \mathcal{F}[u] = \int dt \sqrt{P(t)} \exp[i\phi(t) - i\omega t] \) and \( H(x) \) is the Heaviside function. That is the spectrum is flat-top and truncated at \( \pm \Delta \). The latter parameter plays a role of the spectrum half-width. From Eq. (5) the CSP energy is

\[
E \equiv \int_{-\infty}^{\infty} dt \int_{-\Delta}^{\Delta} \frac{d\omega}{2\pi} P = \frac{6\beta \Delta}{(1 + c) \kappa}. \quad (6)
\]

Important features of (4) are i) \( 0 < c < 2 \), ii) there is no physically nontrivial limit \( \sigma = 0 \), and iii) spectral width increases with \( \sigma \) and \( c \). Hereinafter, the solutions with properties ii) and iii) will be termed as the Schrödinger (or negative) branch of CSP.

### 2.2. Cubic-quintic nonlinear CGLE \((\chi = 0)\)

Let’s consider the case with negligible higher-order SPM and with the SAM in the form of

\[
f\left(|u|^2\right) = \kappa \left( 1 - \zeta |u|^2 \right)|u|^2. \quad (7)
\]

Here the \( \zeta \)-parameter corresponds to the SAM saturation. Such a form of SAM can be attributed to the Kerr-lens mode locking or the mode-locking due to polarization modulator. The former technique uses power-dependence of the laser beam size due to self-focusing inside a nonlinear medium. As a result, the overlapping between the laser and pump beams becomes power-dependent, as well. This leads to the SAM in the form under consideration. For the mode-locking technique utilizing the power-dependent polarization, Eq. (7) can be considered as a low-order in \( P \) approximation of the trigonometric SAM function. The SPM saturation (i.e. the \( \chi \)-term) can be omitted if the beam confocal length inside an active crystal is much less than the crystal length (in a solid-state oscillator) or the mode is strongly confined (in a fiber oscillator). Also, \( \chi = 0 \) for any airless thin-disk oscillator.

Substitution of (3) in (1) supplemented with (7) and taking into account the approximation under consideration result in\textsuperscript{22}
\[ \gamma P = q - \beta \Omega^2, \]
\[ \beta \left( \Omega_t + \frac{\Omega}{P_t} P_t \right) = \kappa P (1 - \zeta P) - \sigma - \alpha \Omega^2, \]

where \( \Omega \equiv \phi_t \) is the frequency deviation from \( \omega_0 \).

The restrictions \( P > 0 \) and \( \Omega_t < \infty \) allow obtaining

\[ \gamma P (0) = \beta \Delta^2 = \frac{3 \gamma}{4 \zeta} \left( 1 - c/2 \pm \sqrt{(1 - c/2)^2 - 4 \zeta / \kappa} \right), \]
\[ \Omega_t = \frac{\beta \zeta \kappa}{3 \gamma^2} (\Delta^2 - \Omega^2) \left( \Omega^2 + \Xi^2 \right), \]
\[ \beta \Xi^2 = \frac{\gamma}{\zeta} (1 + c) - \frac{5}{3} \gamma P (0). \]

Integration of second Eq. (9) in the combination with first Eq. (8) gives the implicit expression for CSP profile.\(^2\)

To define the spectral shape of CSP and its energy, one may use the approach described in previous Subsection. The spectral power is

\[ p(\omega) \approx \frac{6 \pi \gamma H (\Delta^2 - \omega^2)}{\zeta \kappa - \Xi^2 + \omega^2}. \]

That is the CSP spectrum has the truncated Lorentz profile, where \( \Delta \) plays a role of the spectral half-width (if \( \Xi > \Delta \), otherwise the spectral width is defined by \( \Xi \)).

Eq. (10) allows obtaining the CSP energy

\[ E = \frac{6 \gamma}{\zeta \kappa \Xi} \arctan \left( \frac{\Delta}{\Xi} \right). \]

Hence, Eqs. (9,10,11) define the CSP completely. One can see, that the CSP parameters depend on only two control parameters: \( c \equiv \alpha \gamma / \beta \kappa \) and \( a \equiv \sigma \zeta / \kappa \). Since the saturated net-gain is energy-dependent due to gain saturation, one may choose \( c \) and \( E^* \) as two control parameter, where \( E^* \) is the energy of steady-state (continuous-wave, CW) solution of linearized Eq. (1).

Thus, the CSP characteristics can be mapped on two-dimensional diagram (so-called “master diagram”), that makes theirs easily traceable (see below).

2.3. Cubic-quintic nonlinear CGLE (\( \chi \neq 0 \))

Eqs. (1,7) with \( \chi \neq 0 \) describe an oscillator operating under condition, that the beam does not confined strongly in a nonlinear element and its size variation affects the effective SPM. For instance, that can be a solid-state oscillator with the beam confocal length approaching or exceeding the crystal length; a high-energy oscillator with nonlinear plate providing the SPM; a fiber oscillator with no strong mode confinement, etc.

The method sketched in previous Subsections leads to

\[ P = \frac{b}{2} [\Psi - 1], \]
\[ \Delta^2 = \frac{c}{16 \left(1 + \frac{c}{b}\right)} \left[ 2 \left(\frac{3 + \frac{4}{c} + \frac{5}{b}}{1 + \frac{c}{b}}\right) \left(\frac{2 + c}{2} + \frac{3b}{2} \pm \sqrt{(2 - c)^2 - 16a \left(1 + \frac{c}{b}\right)}\right) - 3c - 9b - \frac{32a}{b} - 12 \right], \]
\[ \Omega_t = -\frac{c \psi (1 - \Psi) \left( a + \Omega^2 + \frac{\Omega^2}{4} \left(1 - \Psi\right) \left(1 - \Psi + \frac{\Omega^2}{4} \right) \right)}{\Psi (1 - \Psi) + 4 \Omega^2 / c b}, \]

Proc. of SPIE Vol. 7354  73540T-4
Figure 1. Positive branch: CSP profiles $P(t)$ (solid and dotted curves) and corresponding chirps $\Omega(t)$ (dashed curves and crosses) for the different values of quintic term $b$. $a = 0.01$, $c = 1$.

where $\Psi(t) = \sqrt{1 + \frac{4}{3b}(\Delta^2 - \Omega^2(t))}$ and the following normalizations are used: $t' = t\kappa\sqrt{\kappa/\alpha_\zeta}/\zeta$, $P' = \zeta P$, $\Omega'^2 = \Omega^2\alpha_\zeta/\kappa$, $\Delta'^2 = \Delta^2\alpha_\zeta/\kappa$ (the primes are omitted below). Eqs. (12) represent only solutions, which tend to those of Eqs. (9) when $\chi \to 0$.

Expression for the dimensionless spectral power is

$$p(\omega) \simeq \frac{\pi (A - 1) ((A - 1) cb + 4 (2\omega^2 - \Delta^2))}{cA ((A - 1)(c(a + b + b^2 + \omega^2) + b(\Delta^2 - \omega^2)) - 2(b + 1)(\Delta^2 - \omega^2))} \ rac{H(\Delta^2 - \omega^2)}{}.$$  

(13)

One can see, that the CSP becomes three-parametric due to an appearance of non-zero $\chi$ (i.e. $b \neq \infty$).

The CSPs under consideration subdivide into two classes accordingly two signs in Eqs. (9,12): i) positive branch and ii) negative (or Schrödinger) branch. The negative branch can be transformed into the soliton-like solution of dissipative nonlinear Schrödinger equation, when the quintic nonlinear terms tend to zero (see Subsection 2.1). The positive branch, possessing physically nontrivial limit $a = 0$, does not allow such a transformation. The additional characteristics of these two branches will be described in detail below.

Let’s consider the CSP profiles obtained from integration of (12). Figs. 1 and 2 show the profiles of the positive and negative branches, respectively. The $b$-parameter is scalable and zero quintic SPM corresponds to $b \to \pm \infty$. When $b > 0$, nonlinear phase shift increases with power. Such a sign of $b$ corresponds, for instance, to a Kerr-lens mode-locked oscillator. One can see, that the peak power and, correspondingly, the chirp increases (decreases) with $b$ for positive (negative) branch. When $b < 0$ (saturable SPM), both flat-top (gray solid curve, Fig. 1) and parabolic (dotted curve, Fig. 1) profiles appear.

As it has been pointed (see Eqs. (5,10,13)), the CSP spectra are truncated at some frequency $\pm \Delta$. The spectral profiles of positive branch are shown in Fig. 3. One can see, that there exist next spectral types: i) parabolic-top22,27 (solid curve and circles; the last corresponds to large contribution of saturable SPM), ii) finger-like22,27 (dotted curve; that is the truncated Lorentz profile), and iii) concave27 (dashed curve). In contrast to the model presented in Ref. 29 our model predicts that the concave spectra, which are widely presented in the fiber oscillators, are stable. The last conclusion means that such spectra exist for the saturable SAM (i.e. for $\zeta > 0$). The cause of concave spectra is the positive quintic SPM (i.e. the SPM growing with $P$).

Spectra of negative branch are shown in Fig. 4. The spectra are typically narrower than those of positive branch. Therefore, the finger-like profiles disappear and the typical shapes are i) parabolic- or flat-top, and ii)
**Figure 2.** Negative branch: CSP profiles \( P(t) \) (solid curves) and corresponding chirps \( \Omega_t(t) \) (dashed curves) for the different values of quintic term \( b \). \( a = 0.01, c = 1 \).

**Figure 3.** Positive branch: CSP spectra for the different values of quintic term \( b \). \( a = 0.01, c = 1 \).
Figure 4. Negative branch: CSP spectra for the different values of quintic term $b$. $a = 0.01$, $c = 1$. Profiles for $b = 20$ and -5 coincide.

Figure 5. Normalized spectral half-width in dependence on the saturated net-loss parameter $a$ for the different values of quintic SPM. $c = 1$. Positive branch - solid curves; negative branch - dashed curves.

cave. The latter shape correspond to wider spectra formed in the presence of positive quintic SPM (i.e. the SPM growing with $P$).

Dependence of the normalized spectral half-width $\Delta$ on the normalized saturated net-loss parameter $a$ for the different $b$ is shown in Fig. 5. When $b > 0$ or $b < -2$, the positive branch CSP has a wider spectrum than the negative branch one. Under these conditions, the spectrum width of positive (negative) branch decreases (increases) with the $a$-growth. The behavior of negative branch corresponds to that of CSP in the dissipative nonlinear Schrödinger equation (see Subsection 2.1).

The CSP exists within confined region of $a$, which narrows with the decreasing positive $b$. When $0 > b \geq -2$, the positive branch exists only for nonzero $a$. This means that only negative-branch CSP can spontaneously develop in an oscillator with $0 > b \geq -2$.

Figs. 6,7 show the master diagrams, which are two-dimensional slices of three-dimensional parametric space of CSP (for $\chi \neq 0$). The dimensionless parameter $E$ equals to the dimensional energy $E$ multiplied by $(\kappa/\gamma) \sqrt{\kappa/\alpha}$.
Figure 6. Master diagram: $b \longrightarrow \infty$ (solid curves), $b = 0.2$ (dashed curves). Black curves are the stability thresholds ($a = 0$), gray curves are the border between the positive (+) and negative (−) branches, light gray curves are the positive and negative branches for $a = 0.1$.

and can be easily related to $E^*$ by means of expansion of $\sigma$ in the vicinity of laser threshold$^{22}$: $\sigma \approx \delta (E/E^* - 1) \quad (\delta \equiv \sigma_{E|E=E^*})$. The black curves (the solid ones in both Figs. and the black dashed one in Fig. 6) correspond to the stability threshold against continuum excitation, that is $a = 0$. The stable CSP exists below these curves (i.e., the curves correspond to the maximum values of $c$ for a given $E$). One can see, that the stability threshold $c$ decreases with $E$. Physically, that means, for instance, the growth of GDD required for CSP stabilization ($c \propto 1/\beta$). It is important, that there is an asymptotic value of $c$ (i.e., an extra-growth of $E$ does not change the threshold value of $c$ substantially). The contribution of positive (negative) quintic SPM narrows (broadens) the stability region. Also, the region, where the positive branch exists, narrows for the growing positive $\chi$ (i.e., with the decrease of positive $b$). It should be noted, that the positive branch disappears for small $a$, when $0 > b \geq -2$.

The master diagrams demonstrate main difference between the CSP branches. When $b > 0$ is not too small, the asymptotic behavior of isogains (i.e., the curves of constant $a$) with the $E$-growth demonstrates that the CSP energy is scalable. That is $E \propto E^*$ and the proportionality coefficient is weakly dependent on the parameters of (1). In this sense, the negative (Schrödinger) branch is not energy-scalable, because the energy depends on $c$ weakly and the change of $E$ requires a substantial change of $c$ (e.g., a substantial GDD growth as $c \propto 1/\beta$).

2.4. Generalized nonlinear CGLE

In this subsection the SAM corresponding to a perfectly saturable absorber$^{27}$ will be considered. This type of SAM represents a semiconductor saturable absorber mirror (SESAM), which are extensively used in CPO oscillators.$^{12-15}$ If the CSP width (usually, few picoseconds) exceeds the SESAM relaxation time $T_r$ (hundreds of femtoseconds), the SAM function can be written in the form

$$f \left( |u|^2 \right) = \frac{\kappa |u|^2}{1 + \varsigma |u|^2}, \quad (14)$$

where $\kappa = \kappa_0$, $\mu$ is the modulation depth, $\varsigma = T_r/E_s S$ is the inverse saturation power ($E_s$ is the SESAM saturation energy fluency, $S$ is the beam area on SESAM).

Eqs. (1,14) can be reduced by the above described method to
Figure 7. Master diagram: \( b = -5 \) (solid curves), \( b = -1.8 \) (labeled gray curve). Black curve is the stability threshold \((a = 0)\), gray curve is the border between the positive (+) and negative (−) branches, light gray curves are the positive and negative branches for \( a = 0.1 \). Gray curve \( b = -1.8 \) corresponds to the negative branch.

\[
\zeta P(0) = \frac{\alpha \Delta^2}{\mu c} = \frac{3}{4c} \left( 2 (1 - a) - c \pm \sqrt{\gamma} \right), \tag{15}
\]
\[
\Omega_t = \frac{\alpha}{3 \beta} \frac{\Delta^2 - \Omega^2}{\Delta^2 - \Omega^2 + \gamma / \zeta \beta},
\]
\[
\frac{\alpha \zeta^2}{\mu} = \frac{2 \alpha}{3 \mu} \Delta^2 + 1 - a + c.
\]

Here \( a \equiv \sigma / \mu \), \( c \equiv \alpha \gamma / \beta \kappa \), \( \gamma \equiv (2 - c)^2 - 4a (2 - a + c) \). The spectral profile (truncated parabolic- or flat-top) and the CSP energy are

\[
p(\omega) \approx \frac{6 \pi \beta^2}{\alpha \gamma} \frac{\Delta^2 - \omega^2 + \gamma / \zeta \beta}{\Xi^2 - \omega^2} H (\Delta^2 - \omega^2), \tag{16}
\]
\[
E \approx \frac{6 \beta^2 \Delta}{\alpha \gamma} \left[ 1 - \frac{(\Xi^2 - \Delta^2 - \gamma / \beta \zeta) \arctan \left( \frac{\Delta}{\Xi} \right)}{\Delta \Xi} \right]. \tag{17}
\]

Using the normalizations from previous Subsection reduces Eq. (17) to

\[
E = \frac{6 \Delta}{c^2} \left[ 1 - \frac{(\Xi^2 - \Delta^2 - c) \arctan \left( \frac{\Delta}{\Xi} \right)}{\Delta \Xi} \right], \tag{18}
\]

where primes for the normalized values are omitted.

The master diagram following from Eq. (18) is presented in Fig. 8. The structure of diagram is similar to that described in previous Subsection, but there are two important differences: i) \( c \)-parameter asymptotically tends to zero with growth of \( E \), ii) negative branch is not energy-scalable within all range of existence. The latter conclusion is obvious from behavior of isogains in Fig. 8 as one can see that the \( c \)-scaling weakly affects \( E \). Thus, the energy remains almost constant along this isogain when the GDD scales \((c \propto 1/\beta)\). However, the
energy scaling for the negative branch can be provided by simultaneous growth of $\alpha$ and $\beta$ so that $c$ remains constant (see also the normalization for $E$).

For the negative branch the spectrum narrows with the $c$-decrease due to growth of the GDD contribution (Fig. 9), which stretches the pulse when the energy remains almost constant. When $E$ changes weakly along the isogain corresponding to the negative branch, the spectrum broadens with $\alpha$ (Fig. 9; $c \propto \alpha$). The explanation is that the growth of spectral filtering enhances the cutoff of red (blue)-shifted spectral components located on the pulse front (tail). The growth of cutoff shortens the CSP and, for a fixed energy, $P(0)$ increases. Since $\Delta^2 \propto P(0)$, the spectrum broadens. For the positive branch, the spectrum initially broadens with $c$-decrease (Fig. 9). This can be explained as a result of $E$-increase, which is necessary for keeping in the isogain (Fig. 8). That is, the SPM contribution increases and the spectrum broadens. However, further decrease of $c$ narrows the spectrum due to growth of the CSP width. The latter results from either GDD growth ($c \propto 1/\beta$) or suppression of spectral cutoff on the pulse wings due to $\alpha$-decrease ($c \propto \alpha$).

The CSP spectral width behaves similarly for the cubic-quintic nonlinear CGLE with $\chi = 0$. It should be noted additionally, that the positive branch remains energy-scalable for the SAM under consideration, but there is a need in the decrease of $c$ to provide such a scaling (Fig. 8).

One has to emphasize that the master diagrams cover both solid-state and fiber oscillators. In a fiber oscillator, the GDD value is larger in comparison with that in a solid-state CPO, but the spectral filter bandwidth ($\sim 25$ nm) is small. As a result, the excess of the ratio $\beta/\alpha$ over that for a Ti:sapphire CPO is only tenfold.\textsuperscript{30} Simultaneously, an excess of the ratio $\gamma/\kappa$ over that for a Ti:sapphire CPO is tenfold as well. As a result, the $c$-parameter is $\approx 1$ and, dynamically, there is no substantial distinction in kind between the ANDi fiber and the solid-state CPOs. One difference is that a larger GDD and a comparatively smaller $E$ shift the operational point of a fiber oscillator into the negative branch region. While the operational point of a solid-state oscillator belongs to the positive branch. As a result, their scaling properties are different.

In the case of narrow-band thin disk solid-state oscillator, the assumption $\beta \gg \alpha$ is violated. The result is the smoothed spectrum edges. Such smoothing increases with the decrease of SPM (e.g., when a resonator becomes airless). Nevertheless, the numerical simulations demonstrate that the analytical model describes a spectral width of CSP adequately even for $\alpha \approx \beta$. However, the case of $\alpha > \beta$ is beyond the bounds of the theory in question.
Figure 9. Normalized spectral half-width in dependence on $c$ for the different values of $a$. Positive branch - solid curves; negative branch - dashed curves.

3. CONCLUSIONS

The completely analytical theory of CPOs has been developed. It has been found, that such oscillators (both solid-state and fiber) can be modelled by the nonlinear CGLE, which has been integrated approximately on the basis of the proposed method. As a result, one may easily trace the properties of CSPs, because the CSP has two- (maximum three-)dimensional representation (so-called, master diagram). Four types of SAM have been analyzed on the common basis. These SAMs are inherent in both solid-state and fiber CPOs mode-locked by either self-focusing, or SESAM, or polarization modulator. The CSPs have been found to be subdivided into two classes with different scaling properties. This properties have been traced and described within the whole parametric space.

The mathematical apparatus is presented in detail in http://info.tuwien.ac.at/kalashnikov/NCGLE1.html and http://info.tuwien.ac.at/kalashnikov/genNCGLE.html.

ACKNOWLEDGMENTS

The work was supported by Austrian Fonds zur Förderung der wissenschaftlichen Forschung (project P20293).

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