More clones from ideals

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Outline

1. Background
2. Precomplete clones
3. Fixpoint clones
4. Ideal clones
5. Growth clones
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Clones

We consider \textbf{Algebras} \((X, f, g, \ldots)\) on a \textbf{fixed set} \(X\), and rank them according to their richness of term functions.

\textbf{Note:} In general, our algebras will have many operations.

\textbf{Example}

\[(\mathbb{Q}, +) < (\mathbb{Q}, +, \cdot) < (\mathbb{Q}, +, -, \cdot) = (\mathbb{Q}, -, \cdot).\]

- \textbf{general problem:} Analyse the relationships between different algebras on the same set; by how much is \((\mathbb{Q}, +, \cdot)\) “richer” than \((\mathbb{Q}, +)\)?
- \textbf{specific problem:} Which algebras are \textit{complete}? (i.e., all functions are term functions)?
- Which are \textit{precomplete}? (i.e., will become complete when adding any new function)
Definition

Fix a set $X$. We write $\mathcal{O}^{(n)}$ for the set of $n$-ary operations: $\mathcal{O}^{(n)} = X^{X^n}$, and we let $\mathcal{O} = \mathcal{O}_X = \bigcup_{n=1,2,...} \mathcal{O}^{(n)}$.

A clone on $X$ is a set $C \subseteq \mathcal{O}$ which contains all the projection functions and is closed under composition.

Equivalently, a clone is the set of term functions of some universal algebra on $X$.

Fact

The set of clones on $X$ forms a complete lattice: $\text{CLONE}(X)$.

Definition: For any $C \subseteq \mathcal{O}$ let $\langle C \rangle$ be the clone generated by $C$. We write $C(f)$ for $\langle C \cup \{f\} \rangle$. 
Size of $\text{CLONE}(X)$

If $X$ is finite, then $\emptyset_X$ is countable.

- If $|X| = 1$, then $\emptyset_X$ is trivial.
- If $|X| = 2$, then $\text{CLONE}(X)$ is countable, and completely understood. (“Post’s Lattice”)
- If $3 \leq |X| < \aleph_0$, then $|\text{CLONE}(X)| = 2^{\aleph_0}$, and not well understood.

If $X$ is infinite, then

- $|\emptyset_X| = 2^{|X|}$,
- $|\text{CLONE}(X)| = 2^{2^{|X|}}$, (we will see many proofs of this fact)
- and only little is known about the structure of $\text{CLONE}(X)$. 
Completeness

Example

The functions $\land, \lor, \text{true, false}$ do not generate all operations on $\{\text{true, false}\}$.  

Proof: All these functions are monotone, and $\neg$ is not.

Now let $X$ be any set.

Example

Assume that $\leq$ is a nontrivial partial order on $X$, and that all functions in $C \subseteq O$ are monotone with respect to $\leq$. Then $\langle C \rangle \neq \emptyset$. 


Polymorphisms

Let $X$ be a set, $C \subseteq \mathcal{O}_X$.

- If all functions in $C$ respect some order $\leq$ on $X$,
- or: if all functions in $C$ respect some nontrivial equivalence relation $\theta$
- or: if all functions in $C$ respect some nontrivial fixed set $A \subset X$
  (i.e., $f[A^k] \subseteq A$)
- or . . .

then $\langle C \rangle \neq \emptyset$.

We write $\text{Pol}(\leq), \text{Pol}(\theta), \text{Pol}(A), \ldots$ for the clone of all functions respecting $\leq, \theta, A, \ldots$

Instead of unary $(A)$ or binary $(\leq, \theta)$ relations, we may also consider $n$-ary or even infinitary relations.
**Pol( ) and precomplete clones**

- Every set of the form $\text{Pol}(A_1) \cap \text{Pol}(A_2) \cap \text{Pol}(\theta_3) \cap \cdots$ is a clone.
- Conversely, every clone is the intersection of (at most countably many) sets of the form $\text{Pol}(R)$.

The “maximal” or “precomplete” clones are the coatoms in the clone lattice.

$C \neq \emptyset$ is precomplete iff $C(f) = \emptyset$ for all $f \in \emptyset \setminus C$.

- Every precomplete clone is of the form $\text{Pol}(R)$ for some relation $R$.

**Question**

*Which relations $R$ give rise to precomplete clones?*

This is nontrivial, already for binary relations.
Precomplete clones on finite sets

Example

Let $\emptyset \neq A \neq X$. Then $\text{Pol}(A)$ is precomplete.

Example

Let $X$ be finite. Let $\theta$ be a nontrivial equivalence relation. Then $\text{Pol}(\theta)$ is precomplete.

Theorem (Rosenberg, 1970)

There is an explicit list of all (finitely many, depending on the cardinality of $X$) relations $R$ such that $\text{Pol}(R)$ is precomplete.

Moreover, every clone $C \neq \emptyset$ is below some precomplete clone. This gives an effective (but not efficient) method for checking $\langle D \rangle = \emptyset$, for all $D \subseteq \emptyset$. (Check $f \in D \Rightarrow f \in \text{Pol}(R)$, for all relevant $R$.)
Precomplete clones on infinite sets

Example

Let $\emptyset \neq A \neq X$. Then $\text{Pol}(A)$ is precomplete.

Example

Let $\theta$ be a nontrivial equivalence relation with finitely many classes. Then $\text{Pol}(\theta)$ is precomplete.

For which $R$ is $\text{Pol}(R)$ precomplete? Is every $C \neq \emptyset$ below some precomplete clone?
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Fixpoint clones

Definition

Let $A \subseteq X$. $\text{fix}(A)$ is the set of all functions $f$ satisfying
\[
\forall x \in A : f(x, \ldots, x) = x.
\]
This is a clone.

Definition

Let $F$ be a filter on $X$. $\text{fix}((F))$ is defined as $\bigcup_{A \in F} \text{fix}(A)$, i.e.,
\[
\text{fix}((F)) = \{ g : \exists A \in F \forall x \in A : g(x, \ldots, x) = x \}
\]

- $\text{fix}((F))$ is a clone.
- If $F$ is the principal filter generated by the set $A$, then
  $\text{fix}((F)) = \text{fix}(A)$.
- larger filter $\Rightarrow$ larger clone.
- maximal filter $\Rightarrow$ maximal clone.
Fixpoint clones, application

Let $C_0 := \text{fix}(X)$, i.e. the clone of all functions $f$ satisfying $f(x, \ldots, x) = x$ for all $x \in X$. Let $C_1 := \text{fix}(\emptyset) = O$, the clone of all functions. Then the interval $[C_0, C_1]$ in the clone lattice is rather complicated, and yet we can “explicitly” describe it.

**Theorem (Goldstern-Shelah, 2004)**

The clones in the interval $[C_0, C_1]$ are exactly the clones $\text{fix}((F))$, for all possible filters (including the trivial filter $\mathcal{P}(X)$). (Maximal=precomplete clones correspond to ultrafilters.)

So this interval is order isomorphic to the lattice of closed subsets of $\beta X$ (with reverse inclusion). (Leave it to Topologists . . .

. . . who work on Boolean spaces.)
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Clones from ideals

**Definition**

Let $I$ be a nontrivial ideal on the set $X$ containing all small sets. $f : X^k \to X$ preserves $I$ if $\forall A \in I : f[A^k] \in I$. We write $\text{Pol}(I)$ for the set of all functions preserving $I$.

- $\text{Pol}(I)$ is a clone.
- If $I$ is the principal ideal generated by the set $A$, then $\text{Pol}(I) = \text{Pol}(A)$.
- larger ideal $\not\Rightarrow$ larger clone.
- maximal ideal $\Rightarrow$ maximal clone.
- However, many other ideals also yield maximal clones. $I^{-\circ} := \{ A \subseteq X : \forall B \in [A]^\omega : [B]^\omega \cap I \neq \emptyset \}$.
  If $I = I^{-\circ}$, then $\text{Pol}(I)$ is maximal.
Ideal clones, application

Let $X := 2^{<\omega}$, the full binary tree. Every $\eta \in 2^\omega$ defines a branch $b_\eta = \{\eta \upharpoonright n : n \in \omega\}$ through this tree.

For every subset $A \subseteq 2^\omega$ we define an ideal $I_A$:

$$I_A = \{E \subseteq 2^{<\omega} : \forall \eta \in A \; |b_\eta \cap E| < \aleph_0\}$$

Easy to check that $I_A = I_A^{\circ}$, and that the ideals $I_A$ are all different.

Theorem (Beiglböck-Goldstern-Heindorf-Pinsker, 2007)

While the ideals $I_A$ are not maximal, the clones $\text{Pol}(I_A)$ are (for nontrivial $A$).

This gives an explicit example of $2^\mathfrak{c}$ many precomplete clones on a countable set. (Even without AC.)

Question

Find such examples on uncountable sets.
Equivalence relations

Example

Let $\theta$ be a nontrivial equivalence relation on a finite set. Then $\text{Pol}(\theta)$ is a precomplete clone.

Example

Let $\theta$ be a nontrivial equivalence relation on any set, with finitely many classes. Then $\text{Pol}(\theta)$ is a precomplete clone.
Definition

Let $\mathcal{E}$ be a directed family of equivalence relations (coarser and coarser).
Define $\text{Pol}(\mathcal{E})$ as the set of all functions $f : X^k \rightarrow X$ with:
for all $E \in \mathcal{E}$ there is $E' \in \mathcal{E}$ such that: whenever $\vec{x} E \vec{y}$, then $f(\vec{x}) E' f(\vec{y})$.

When is $\text{Pol}(\mathcal{E})$ precomplete? Difficult. Because...

Fact

For every ideal $I$ there is a family $\mathcal{E}$ as above such that $\text{Pol}(I) = \text{Pol}(\mathcal{E})$. 
Outline

1. Background
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Growth clones

**Definition**

Let $X = \omega = \{0, 1, 2, \ldots\}$ for simplicity. For every infinite $A = \{a_0 < a_1 < \cdots\} \subseteq X$ we define $\text{bound}(A)$ as the set of functions which do not jump to far in $A$:

$$\text{bound}(A) := \{f : \exists k \forall i : x < a_i \Rightarrow f(x) < a_{i+k}\}$$

(This is a clone.)

A similar construction is possible for uncountable Sets.
Let $X = \omega$ again. For every filter $F$ of subsets of $X$ we define
\[ \text{bound}(\mathcal{F}) := \bigcup_{A \in \mathcal{F}} \text{bound}(A). \]

\[ \text{bound}(\mathcal{F}) := \{ f : \exists A \in \mathcal{F} \exists k \forall i : x < a^A_i \Rightarrow f(x) < a^A_{i+k} \} \]

(\text{where } a^A_0 < a^A_1 < \cdots \text{ is the increasing enumeration of } A).

- bound($\mathcal{F}$) is a clone.
- If $F$ is the principal filter generated by the set $A$, then $\text{bound}(\mathcal{F}) = \text{bound}(A)$.
- larger filter $\Rightarrow$ larger clone.
- maximal filter $\not\Rightarrow$ maximal clone.
- (In fact, $\text{bound}(\mathcal{F})$ is never a maximal clone.)
Theorem

\textbf{G*-Shelah, 2006}

Assume GCH. Then on every infinite set \(X\) there is a filter \(F\) such that, letting \(C := \text{bound}(F)\), we know the interval \([C, \emptyset)\) quite well, and it is (more or less) a quite saturated linear order with no last element.

In particular: not every clone is below a precomplete clone.