Credit Risk and Dynamic Capital Structure Choice

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Abstract

This paper presents an analysis of the effect of dynamic capital structure adjustments on credit risk. Firms may optimally adjust their leverage in response to stochastic changes in firm value. This is shown to influence a bond’s expected default frequency and its fair credit spread. Generally capital structure dynamics significantly increase both credit spreads and expected default probabilities. Numerical examples demonstrate that there exists a u-shaped relationship between the traditional distance to default measure and expected default frequencies. The magnitude of the effect of capital structure dynamics is shown to depend on firm characteristics such as asset volatility, the growth rate, the effective corporate tax rate, call features and transactions costs. The results therefore suggest a cross-sectional variation of the relationship between the distance to default and expected default frequencies.

Finally we extend the analysis to include the estimation of the firm’s asset value and its volatility from observed equity prices. We find that the underestimation of credit spreads and expected default frequencies is exacerbated when the asset value and volatility are inferred from a model which ignores the opportunity to recapitalize.
1 Introduction

Measuring and managing credit risk has become of central importance for financial institutions. In most countries, banks’ equity requirements are already tied to their exposure to credit risk. According to the proposed Basel Accord II, the link between credit risk and capital requirement will be regulated in much more detail. Banks will be allowed to calculate their credit risk exposure and thus their equity requirements on the basis of their internal rating models.

Perhaps even more importantly, the search for shareholder value requires that banks can accurately quantify their exposures to unexpected credit losses. This is a prerequisite for a correct allocation of economic capital to various lending activities and thus for optimizing the capital budgeting decisions.

Despite their importance for regulation and the management of financial institutions, existing credit risk models are still unable to capture some important risk factors. For example, most existing credit risk models assume that the firm’s debt level remains constant over time or changes in a deterministic way. In practice firms’ adjust their financial structures in response to stochastic changes in their economic environment. This may have significant influence on credit risk.

In this paper we show how firms’ dynamic capital structure choices can be integrated into a credit risk model. We analyze the effect of intertemporal capital structure choices on a corporate bond’s fair credit spread, on estimated distances to default, and on expected default frequencies.

We present a model where the firm’s free cash flow follows a geometric Brownian motion. This cash flow is partly used to pay the coupon on the firm’s debt and the remainder is paid out as a dividend to equity holders.

Debt is advantageous for tax reasons. The net tax advantage of debt is the
difference between the corporate tax advantage of debt (interest is corporate tax deductible) and the personal tax disadvantage of debt (interest income is taxed more heavily than capital gains or dividends).  

Recapitalizations are associated with transactions costs. As a result firms do not adjust their capital structures continually. If the free cash flow increases by a sufficient amount, then the firm may find it optimal to issue more debt. Since the risk free rate of interest is assumed constant and since the new optimal leverage ratio is equal to the initially chosen leverage ratio the new debt can be issued at precisely the same terms as the original debt.

We contrast our model of dynamic recapitalization with the traditional approach in the spirit of Merton (1974) where the face value of debt at the risk horizon is assumed to be fixed. We find that consideration of dynamic recapitalization decisions generally increases fair credit spreads and the expected default frequencies. Interestingly we find a non-monotonic, u-shaped relationship between distance to default and expected default frequencies. One of the major implications of our analysis is that it would be wrong to estimated an unconditional relationship between distance to default and expected default frequencies. Our results indicate that one must condition on the firm’s asset volatility, its effective corporate tax rate, its expected growth rate and estimated bankruptcy costs.

Our model is related to several papers. As in Fischer, Heinkel, and Zechner (1989) we explicitly model the possibility of dynamic capital structure changes. We extend the analysis to focus on the impact of dynamic capital structure adjustments on fair credit spreads and expected default frequencies. Also, we use the

1Interest is taxable at the personal level whereas the realized rate of return on equity is not. This is so since we assume that the rate of return on equity is either realized in the form of tax free capital gains or realized in the form of dividends which are not taxed because of imputation of the corporate tax rate.
firm’s cash flow as the state variable, rather than the value of the firm’s unlevered assets, as in Fischer, Heinkel, and Zechner (1989).

Christensen, Flor, Lando, and Miltersen (2000) develop a model of dynamic capital structure adjustments. They explicitly explore the impact of renegotiations between equity holders and debt holders in times of financial distress whereas we do not allow for renegotiation. The main focus of our paper is the comparative static analysis of the impact of dynamic recapitalization on credit risk.

Collin-Dufresne and Goldstein (2000) analyze whether or not credit spreads reflect stationary leverage ratios. In their model, leverage ratios are mean reverting. Consistent with empirical evidence they find that in comparison to a model with constant leverage debt issued by low-leverage firms has larger credit spreads and that the term structure of debt is upward sloping for low-grade debt.

We extend the analysis of the effects of dynamic leverage adjustments on credit spreads in Collin-Dufresne and Goldstein (2000) by explicitly modeling equityholders’ optimal capital structure choices. This allows us to explore how the effect of capital structure dynamics on credit spreads is related to the characteristics of issuing firms. By modeling capital structure choices endogenously we also recognize that leverage adjustments are asymmetric. A firm increases its debt level when its firm value increases but debt tends to be “sticky” when firm value decreases. This feature of capital structure dynamics influences credit spreads and expected default frequencies.

The remainder of the paper is organized as follows. Section 2 introduces the model. The results of the analysis are presented in Section 3. Section 4 summarizes and concludes.
2 The Model

We assume that the firm’s instantaneous free cash flow after corporate tax $c_t$ follows a geometric Brownian motion given by

$$\frac{dc_t}{c_t} = \mu dt + \sigma dW_t,$$

where the expected drift rate and the instantaneous variance of the flow process are determined by $\mu$ and $c_t^2 \sigma^2$ respectively (see Table 1 for the notation used throughout the paper), $dW_t$ is the increment to a standard Wiener process. Hence, if $r$ and $\hat{\mu}$ denote riskless interest rate and the risk adjusted drift rate of the cash flow process respectively and $\tau_p$ the personal tax rate on ordinary income then the current value of the unlevered flow is given by $\frac{c_t}{r(1-\tau_p)-\hat{\mu}}$.

In our model we assume that the effective corporate tax rate $\tau_c$ exceeds $\tau_p$. Thus, given that coupon payments are tax deductible and considering that issuing debt causes default risk, firms have an incentive to issue debt and to maintain an optimal capital structure. Allowing for dynamic capital structure, $B_t$ denotes the face value of outstanding debt at time $t$, i.e., $B_t$ is endogenously determined by the decision makers within the firm. We define the firm’s inverse leverage ratio $y_t$ as

$$y_t = \frac{1}{B_t} \frac{c_t}{r(1-\tau_p)-\hat{\mu}},$$

However, since we assume that it is costly to call outstanding debt (call premium $\lambda$) as well as to issue new debt (proportional transactions costs $k$) it is not optimal to adjust the capital structure instantaneously. Rather than reacting to any change in the firm’s leverage ratio, only sufficient deviations from the optimum
satisfy the expenses associated with a reorganization of debt (see Fischer, Heinkel, and Zechner (1989)). Consequently, the inverse leverage ratio $y_t$ is driven by the dynamics

$$
\frac{dy_t}{y_t} = \begin{cases} 
\mu dt + \sigma dW_t & : \text{no debt reorganization at time } t, \\
\frac{B_t}{B_t^*} - 1 & : \text{debt is restructured from } B_t \text{ to } B_t^* \text{ at time } t,
\end{cases}
$$

that is, during periods where the amount of debt issued is constant, the inverse leverage ratio follows the same geometric Brownian dynamics as the cash flow process $c_t$. Whenever the firm’s management finds it optimal to reorganize debt, the face value $B_t$ jumps to the amount of newly issued debt and analogously there is a jump in the inverse leverage ratio $y_t$. 

7
Figure 1: One particular realization of $y_t$. When the firm reorganizes its debt (when $y_t$ hits $\bar{y}$) the inverse leverage ratio jumps to $y_0$. This jump reduces the distance of $y_t$ to the critical default threshold $\underline{y}$, and thus, increases the default probability of the firm that dynamically adjusts its capital structure.

In the sequel we interpret the firm’s equity $E$ and debt $D$ as claims contingent on $y_t$ and $B_t$ rather than as claims contingent on the firm’s profit flow $c_t$. This construction allows us to formulate the entire model to be homogenous in $B$, i.e., $E(y,B) = BE(y,1)$ and $D(y,B) = BD(y,1)$. The reason is the fact that both the cash flow $c_t = (r(1-\tau)-(\hat{\mu})y_t)B_t$ and the coupon flow $iB_t$ as well as payments in the case of debt restructuring are proportional to $B_t$. The assumption of proportional bankruptcy costs preserves this homogeneity.

Following Fischer, Heinkel, and Zechner (1989), we consider reorganization strategies determined by an upper threshold $\bar{y}$ and a lower threshold $\underline{y}$ for the in-
verse leverage ratio. This means, whenever $y_t$ reaches $\overline{y}$ the amount of outstanding debt is increased by calling existing debt and issuing new contracts. Whenever $y_t$ reaches $\underline{y}$ equityholders decide to default. As a consequence of the homogeneity it is – in the case of a reorganization – always optimal to establish a certain optimum leverage ratio, denoted by $y_0^*$. This means, the current amount of debt $B_t$ is kept constant as long as $y_t$ is in the range between $\underline{y}$ and $\overline{y}$. Only if $y_t$ hits $\overline{y}$, $B$ jumps to $\frac{\overline{y}}{\overline{y}}B_t$. If equityholders default, the ownership is transferred to the bondholders. After paying bankruptcy costs they will optimally lever the firm, i.e., $y$ will immediately jump to $y_0^*$. Of course, in order to be a consistent reorganization strategy, we require $\underline{y} < y_0^* < \overline{y}$. Figure 1 plots one particular realization of $y_t$ which illustrates the characteristics of the dynamics of the firms inverse leverage ratio. On a reorganization of debt (when $y_t$ hits $\overline{y}$) $y_t$ jumps to $y_0$ thereby reducing the distance to the critical default trigger $\underline{y}$.

2.1 The Value of Equity and Debt

In this subsection we consider a given (not necessarily optimal) reorganization strategy $(\underline{y}, y_0^*, \overline{y})$ and determine the value of equity and debt. Based on these results we subsequently determine the optimal strategy in Subsection 2.2.

Since $B$ is kept constant in the interval $(\underline{y}, \overline{y})$ we can apply standard contingent claims valuation techniques to determine the value of equity $E(y, B)$ and debt $D(y, B)$. More precisely, when the face value of debt issued is constant, $B_t = B$, the value of equity and debt must satisfy

$$\frac{1}{2}\sigma^2 y^2 E_{yy} + \mu y E_y - r(1 - \tau_p)E - (1 - \tau_c)iB + (r(1 - \tau_p) - \hat{\mu}) y_t B = 0, \quad (4)$$

$$\frac{1}{2}\sigma^2 y^2 D_{yy} + \mu y D_y - r(1 - \tau_p)D + (1 - \tau_p)iB = 0. \quad (5)$$
The solutions to these second order ordinary differential equations are

\[
E(\gamma; B) = B E_1 \gamma^{m_1} + B E_2 \gamma^{m_2} - \frac{(1 - \tau_p) i}{(1 - \tau_p) r} B + \gamma_i B, \\
D(\gamma; B) = B D_1 \gamma^{m_1} + B D_2 \gamma^{m_2} + \frac{i}{r} B,
\]

where \(m_1\) and \(m_2\) are the positive and the negative root of the characteristic quadratic polynomial, i.e.,

\[
m_{1,2} = \frac{1}{2} - \frac{\hat{\mu}}{\sigma^2} \pm \sqrt{\left(\frac{1}{2} - \frac{\hat{\mu}}{\sigma^2}\right)^2 + \frac{2r(1 - \tau_p)}{\sigma^2}},
\]

and \(E_{1,2}\) and \(D_{1,2}\) are constants that have to be determined by the following conditions.

\[
E(\gamma; B) = 0, \\
E(\overline{\gamma}; B) = \left[ V(\gamma_0, B \frac{\overline{\gamma}}{\gamma_0}) - kB \frac{\overline{\gamma}}{\gamma_0} \right] - (1 + \lambda) B,
\]

where \(V\) denotes the total value of the firm, \(V = E + D\). Equation (9) states that equity is worthless in the case of default. When the firm is recapitalized (Equation (10)), it first buys back the outstanding debt securities, paying \((1 - \lambda) B\). After that the firm immediately releverages optimally, i.e., it issues new debt with a face value \(\overline{\gamma}\). This is covered by the first term of Equation (10).
The boundary conditions for debt valuation are

\[ D(\bar{y}, B) = \left[ V(y_0, B \frac{y}{y_0}) - kB \frac{y}{y_0} \right] (1 - g), \quad (11) \]
\[ D(\bar{y}, B) = (1 + \lambda)B. \quad (12) \]

On default (Equation (11)) the bondholders become owners of the firm which they immediately re-lever optimally. The proportional bankruptcy costs are borne by the new owners of the firm. When the firm is recapitalized (Equation (12)) the outstanding debt is called back at the price \((1 + \lambda)B\).

Since we assume that debt is always issued at par we determine the coupon rate \(i\) endogenously:

choose \(i\) such that \(D(y_0) = B. \quad (13)\)

### 2.2 Optimal Recapitalization

In the previous subsection we have derived the value of equity and debt under a given recapitalization strategy \((\bar{y}, y_0, \bar{y})\). Now we wish to determine the optimal choice of these critical values. When the firm decides either to recapitalize (at \(\bar{y}\)) or to default (at \(\bar{y}\)), it is a levered firm and, thus, these values are both determined in order to satisfy equityholders. In contrast, after calling debt, the firm is unlevered, and therefore, the amount of newly issued debt is determined from a firm value optimizing point of view. Precisely, for a given \(y_0\) equityholders will optimize their decision variables \(\bar{y} = \bar{y}^*(y_0)\) and \(\bar{y} = \bar{y}^*(y_0)\) which simultaneously satisfy the first order conditions of optimality (see Dixit (1993) for a discussion of the so called ‘smooth pasting’ conditions)
When issuing new debt – thereby fixing the coupon rate \( i \) to issue the bond at par – the owner of the unlevered firm anticipates the recapitalization strategy and chooses the optimal initial capital structure by solving

\[
\max_{y_0} V(y_0, B) - kB,
\]

subject to

\[
B = \frac{c_0}{y_0 r(1 - \tau_p) - \mu}, \quad y = y^*(y_0), \quad \bar{y} = \bar{y}^*(y_0), \quad i : D(y_0, B) = B.
\]

Therefore, the first order condition that has to be satisfied by the optimal initial inverse leverage ratio \( y_0^* \) is

\[
\frac{\partial V}{\partial y}(y_0^*, B) + \frac{\partial V}{\partial y_0}(y_0^*, B) + \frac{\partial V}{\partial \bar{y}} \frac{\partial \bar{y}}{\partial y_0}(y_0^*, B) + \frac{\partial V}{\partial y^*} \frac{\partial y^*}{\partial y_0}(y_0^*, B)
\]

\[
+ \frac{\partial V}{\partial i} \frac{\partial i}{\partial y_0}(y_0^*, B) - \frac{1}{y_0^*} (V(y_0^*, B) - kB) = 0.
\]
2.3 A Benchmark: The Case of a Constant Debt Level

As a benchmark we use the case where the firm is not allowed to reorganize debt, i.e., \( B_t = B_0 \), the initially chosen amount of debt cannot be changed. As a consequence, the set of decision variables contains only the initial capital structure \( y_0 \) and the lower critical value \( y \). The absence of the reorganization opportunity is further reflected in a change in the boundary conditions. Since there exists no upper threshold that triggers a jump in the capital structure the conditions (10) has to be substituted by

\[
\lim_{y \to \infty} E(y, B) = -\frac{(1 - \tau_c)i}{(1 - \tau_p)r}B + y_tB,
\]

and condition (12) has to be substituted by

\[
\lim_{y \to \infty} D(y, B) = \frac{iB}{r}.
\]

Since \( E \) and \( V \) are now independent of \( y \) the optimality condition (15) has to be dropped, and \( \frac{\partial V}{\partial y} = 0 \) can be substituted into condition (17).

3 Results and Comparative Statics

Presenting the results of the model analysis this section is composed of three subsections. Each of them starts from a common base case scenario (see Table 2 for the parameter values) and discusses comparative statics. The first (Subsection 3.1) focuses on the firm’s optimal capital structure choice when it is allowed to dynamically reorganize debt compared to the ‘Merton like’ benchmark model with static
Table 2: Base case parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
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<tr>
<td>riskless rate of interest</td>
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<tr>
<td>personal tax rate ( \tau_p )</td>
<td>35%</td>
</tr>
<tr>
<td>corporate tax rate ( \tau_c )</td>
<td>50%</td>
</tr>
<tr>
<td>variance ( \sigma^2_y )</td>
<td>5%</td>
</tr>
<tr>
<td>risk adjusted drift ( \hat{\mu} )</td>
<td>0%</td>
</tr>
<tr>
<td>transactions costs ( k )</td>
<td>1%</td>
</tr>
<tr>
<td>call premium ( \lambda )</td>
<td>0%</td>
</tr>
<tr>
<td>bankruptcy costs ( g )</td>
<td>25%</td>
</tr>
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</table>

debt level. Since we endogenously determine the firm’s capital structure choice, we are able to explore the impact of firm characteristics (like the variance of the cash flow process, the growth rate, or the tax advantage of debt) on fair credit spreads and the optimal initial leverage ratio. The second (Subsection 3.2) concentrates on model risk from an analyst’s point of view. Specifically, we estimate credit risk from observed equity time series and examine the impact of the model choice on credit spreads. In the third (Subsection 3.3) we determine expected default frequencies implied by a dynamic capital structure choice. Comparing the results to the benchmark model we show that assuming static debt level significantly underestimates expected default frequencies. Furthermore, we discuss the impact of firm characteristics on the relationship between distance to default and expected default frequencies.

3.1 The Firm’s Optimal Capital Structure Choice

How does the option to dynamically adjust the capital structure affect a firm’s financing decisions? Table 3 shows the optimal recapitalization strategy of a firm
with dynamic capital structure together with the optimal choice of a firm with static debt level. The most evident difference is that a dynamic firm initially uses much less debt than the static does. The dynamic firm anticipates the fact that it will increase debt in the case the firm value evolves well. It finds the optimal choice by balancing the tax benefits of debt against the costs of capital (including coupon payments and the costs associated with recapitalization). When the static firm wants to take the full advantage of the tax benefits in the case the firm evolves well it initially has to take a larger amount of debt. Counterintuitively, the fair coupon rate of the dynamic firm exceeds that of the static firm. The reason is the fact that the dynamic firm calls back existing debt and when the firm value increases sufficiently and issues a larger amount of debt. This action increases default risk because it decreases the distance of critical default threshold (see Figure 1). Or, from an other point of view it eliminates the chance for debtholders that the value of their contract grows significantly above par.

Table 4 lists comparative statics on $\sigma^2$, $\tau_c - \tau_p$, $k$, $g$, $\hat{\mu}$, and $\lambda$. We see that the opportunity to recapitalize reduces the optimal initial leverage ratio and that it generally increases credit spreads. These effects are stronger for high-risk firms, for firms with large tax advantage of debt and for high-growth firms. These effects are less pronounced for firms with high costs of recapitalization and for firms with
Table 4: Comparative static analysis

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<th>$1/y_0^*(\text{stat.})$</th>
<th>$i$ (stat.)</th>
<th>$\Delta(1/y_0^*)$</th>
<th>$\Delta i$</th>
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<th>$i$ (stat.)</th>
<th>$\Delta(1/y_0^*)$</th>
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<th>$i$ (dyn.)</th>
<th>$1/y_0^*(\text{stat.})$</th>
<th>$i$ (stat.)</th>
<th>$\Delta(1/y_0^*)$</th>
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<td>7.4%</td>
<td>-6.2%</td>
<td>-10 bp</td>
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</table>

high bankruptcy costs.

### 3.2 Model Risk

In the previous section we have explored the effect of dynamic recapitalization on leverage choice and credit spreads. We have thereby assumed that $y$ and $\sigma_y$ are observable to all parties. For practical credit risk management applications,
however, the total value of the firm’s assets and its volatility and thus $y$ and $\sigma_y$ must be inferred from the observable market value of equity, $E$ and $\sigma_E$. To evaluate the importance of capital structure dynamics on credit risk estimates we therefore extend our analysis to incorporate this estimation step.

We analyze a credit risk manager who observes the market value of equity and its volatility. From this observation she wishes to infer the fair credit spread of a corporate bond. Our benchmark is the static Merton-type model where firms cannot adjust leverage. For a given $E$ and $\sigma_E$ this model is first used to infer $y$ and $\sigma_y$ and then to calculate required credit spreads. We compare this with the results from our dynamic capital structure model. Thus, for the same $E$ and $\sigma_E$ we use the model of Section 2 to infer $y$ and $\sigma_y$ and then to calculate fair credit spreads.

We proceed as follows. For a given set of parameter values we calculate the optimal initial leverage and fair credit spread based on the dynamic capital structure model of Section 2. For this initial leverage ratio this model also generates an equity value $E$ and an equity volatility $\sigma_E$.

In a second step we use these values for $E$ and $\sigma_E$ in the static debt level model (see Subsection 2.3). In particular we use the static model to numerically calculate the $V$ and $\sigma_V$ consistent with the initially generated equity value and volatility. Finally, given these parameters, we use the static model to calculate the resulting credit spread. Table 5 summarizes the results of our numerical examples.

We first note that “consistent” use of the static leverage model to infer the value and the volatility of the underlying and then to calculate credit spreads underestimates the true fair credit spread in all examples. However, the error due to ignoring capital structure dynamics significantly depends on the firm’s characteristics. First, the underestimation of required credit spreads increases with the volatility
of the firm’s cash flows. The more volatile the firm’s underlying cash flows are, the more likely it is that the option to increase leverage is exercised. Thus, for firms in risky industries it is more important to take capital structure dynamics into account.

Second, the underestimation of required credit spreads depends on the degree to which original debtholders are protected from leverage increases. This is captured by the parameter \( \lambda \) in our model. If issuing additional debt requires that the old debtholders receive the face value, then the underestimation of the fair credit spread that results if one ignores leverage adjustments is 71 basis points. By contrast, if existing debt must be repurchased at a premium of 25 percent before a capital structure adjustment can take place, then the error is negligible, i.e. 2 basis points.

Another important parameter that influences the magnitude of the error is the

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Table 5: Comparative static analysis (model risk)

<table>
<thead>
<tr>
<th>( \sigma_i^2 )</th>
<th>( i ) (dyn.)</th>
<th>( i ) (stat.)</th>
<th>( \Delta i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.02</td>
<td>6.35%</td>
<td>6.04%</td>
<td>31 bp</td>
</tr>
<tr>
<td>0.04</td>
<td>7.30%</td>
<td>6.73%</td>
<td>57 bp</td>
</tr>
<tr>
<td>0.06</td>
<td>8.14%</td>
<td>7.34%</td>
<td>80 bp</td>
</tr>
<tr>
<td>0.08</td>
<td>8.85%</td>
<td>7.85%</td>
<td>100 bp</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>( i ) (dyn.)</th>
<th>( i ) (stat.)</th>
<th>( \Delta i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0%</td>
<td>7.75%</td>
<td>7.07%</td>
<td>68 bp</td>
</tr>
<tr>
<td>5%</td>
<td>7.40%</td>
<td>7.06%</td>
<td>34 bp</td>
</tr>
<tr>
<td>10%</td>
<td>7.28%</td>
<td>7.10%</td>
<td>18 bp</td>
</tr>
<tr>
<td>25%</td>
<td>7.22%</td>
<td>7.20%</td>
<td>2 bp</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \tau_c - \tau_p )</th>
<th>( i ) (dyn.)</th>
<th>( i ) (stat.)</th>
<th>( \Delta i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>15%</td>
<td>7.75%</td>
<td>7.07%</td>
<td>68 bp</td>
</tr>
<tr>
<td>11%</td>
<td>7.03%</td>
<td>6.92%</td>
<td>11 bp</td>
</tr>
<tr>
<td>5%</td>
<td>5.97%</td>
<td>5.97%</td>
<td>0 bp</td>
</tr>
</tbody>
</table>
effective tax advantage of debt, i.e. $\tau_c - \tau_p$. If this difference is 15 percent, then the resulting underestimation of credit spreads in a static credit risk model is 71 basis points. If the net tax advantage of debt is reduced to 5 percent, then the error is essentially zero.

### 3.3 Theoretical Expected Default Frequency

In this subsection we examine the impact of a dynamic capital structure on the default probability. We calculate the theoretical expected default frequency (TEDFs) of a firm, which we define as the probability that the firm defaults within a certain time period of length $T$. Since TEDFs are computed with respect to the objective probability measure, it is $\mu$ which determines the drift of the underlying process $y_t$ in the respective calculations. However, when choosing the reorganization thresholds, equityholders will find their optimal decision applying the risk neutral valuation presented in previous sections. This means, the critical thresholds $y$, $y_0$, and $\overline{y}$ are determined with respect to the risk adjusted drift $\hat{\mu}$ while the probability of hitting $y$ (which triggers default) within the period $T$ depends on the objective drift $\mu$. ???

If no recapitalization is allowed, the probability that the firm with initial inverse leverage ratio $y$ does not default within the next $T$ years is given by

$$P_0(y, T) = N\left(\frac{\ln(y/y)+(\mu-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)$$

$$- \left(\frac{y}{\overline{y}}\right)^{\frac{2(\mu-\frac{1}{2}\sigma^2)}{\sigma^2}} N\left(\frac{\ln(y/y)+(\mu-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right),$$

where the subscript 0 indicates, that recapitalization is not allowed (The formu-
lae in this subsection are derived using the results for standard Brownian motion with drift tabulated in Borodin and Salminen (1996)). Therefore, the theoretical expected default frequency in the static case is

\[ \text{TEDF}(y, T) = 1 - P_0(y). \] (21)

For a firm that dynamically adjusts its capital structure using the optimal recapitalization strategy \((y, y_0, \overline{y})\) we proceed in several steps. Assuming that the initial inverse leverage ratio is \(y\), we first calculate the probability that the firm neither defaults nor recapitalizes within the next \(T\) years. This probability of surviving whereby keeping stable capital structure is given by

\[
P_s(y, T) = \sum_{k=-\infty}^{\infty} \left( e^{-2\left(\frac{\ln(y/y_0) + \frac{1}{2}\sigma^2 k}{\sigma^2}\right)[N(d1) - N(d2)]} - e^{-2\left(\frac{\ln(y/y_0) + \frac{1}{2}\sigma^2 k}{\sigma^2}\right)[N(d3) - N(d4)]} \right),
\] (22)

where \(d_1, d_2, d_3,\) and \(d_4\) are determined by

\[
\begin{align*}
d_1 &= \frac{\ln(y/y_0) - 2\ln(\overline{y}/y)k + (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \\
d_2 &= -\frac{\ln(\overline{y}/y) - 2\ln(\overline{y}/y)k + (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \\
d_3 &= -\frac{-\ln(y/y_0) - 2\ln(\overline{y}/y)k + (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \\
d_4 &= -\frac{-2\ln(\overline{y}/y)(1 + k) + (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}.
\end{align*}
\]

and \(N(.)\) denotes the cumulative distribution function of the standard normal dis-
tribution.

In a second step we determine the recapitalization density \( f(y, t) \), i.e., \( 1/dt \) times the probability that a firm with initial inverse capital ratio \( y \) will recapitalize within the time interval \([t, t+dt]\). This is given by

\[
f(y, t) = e^{((\mu - \frac{1}{2}\sigma^2) \ln(y) - (\mu - \frac{1}{2}\sigma^2)^2)}
\times \sum_{k=0}^{\infty} \frac{\ln(y) + 2\ln(y)k}{\sqrt{2\pi t^{2/3}}} e^{-\frac{(\ln(y) + 2\ln(y)k)^2}{2\sigma^2 t}}
\]  

(23)

In a last step we determine the probability that a firm which is allowed to recapitalize \( n \) times will not default within \( T \) years using the iteration rule

\[
P_n(y, T) = P_1(y, T) + \int_0^T P_{n-1}(y_0, T-t) f(y, t) dt
\]  

(24)

Two mutually exclusive events, contribute to this probability. Either the firm survives without recapitalization (represented by the first term). Or it recapitalizes at some time \( t \) and survives another \( T-t \) years starting from \( y_0 \), the second term of Equation (24) integrates over all possible recapitalization times. However, this integration has to be performed numerically. The probability that a firm with dynamic capital structure defaults within the next \( T \) years is therefore

\[
TEDF(y, T) = \lim_{n \to \infty} [TEDF_n(y, T) = 1 - P_n(y, T)].
\]  

(25)

Equation (25) allows to examine the contribution of multiple recapitalization to TEDFs. Our studies show, that this contribution converges very fast to zero. However, to calculate TEDFs over three years, the first three recapitalization options significantly contribute to the default probabilities.
Figure 2: Expected default frequencies with dynamic and static debt level plotted against the distance to default. While static debt level leads to a decreasing relation between DD and TEDF, dynamic capital structure leads to a u-shaped relationship. Debt reorganization at $\bar{y}$ leads to a reduction in the distance to default and therefore increases the default probability when $y$ approaches $\bar{y}$.

In the following we compute TEDFs over a time horizon of three years, i.e., $T = 3$. Figure 2 compares the three year TEDF of a firm (we take the base case parameters from Table 2 and set $\mu = 0$) with dynamic capital structure to that of a firm with static debt level. These default frequencies are plotted against the ‘distance to default’ (DD) which we define with respect to a one period credit risk model as

$$DD(y) = \frac{\ln(y/\bar{y}) + (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}.$$  \hspace{1cm} (26)

That is, in a model where default occurs only at the end of the time horizon $T$. 

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Figure 3: Expected default frequencies with dynamic capital structure and the unconditional probability that default occurs after recapitalization, i.e., the probability that the firm recapitalizes and defaults subsequent to the reorganization. When $y$ is close to the recapitalization threshold $\gamma$, nearly the entire default probability comes from defaults that occur subsequent to a recapitalization.

in the case where the time $T$ inverse leverage ratio $y_T$ is less than $y$ (Merton ?), the probability of default is given by $N(DD(y))$. Or in other words, in a one period credit risk model a distance to default of 1.65 corresponds to a default probability of $N(1.65) = 5\%$ (or $DD = 3$ corresponds to a default probability of 0.135% respectively). Since the benchmark model (see Subsection 2.3) allows for bankruptcy at any moment within the time horizon, the TEDF of a firm with static debt level exceeds $N(DD)$ (see Equation (20)). However, for static debt level TEDF is monotone decreasing in the distance to default. Hence, the larger DD, the lower is the default probability.
Figure 4: The underestimation of TEDFs when ignoring the opportunity to recapitalize plotted against the distance to default for different risk levels. The underestimation is higher for high-risk firms.

When the firm dynamically adjusts its capital structure, the monotonicity of TEDF is lost. Firms that are performing well have a high probability that they will recapitalize in the near future. When recapitalizing, the inverse leverage ratio jumps from $y^*$ to $y_0^*$ (see Figure 1), with the consequence that for firms with dynamic capital structure we have $TEDF(y^*,.) = TEDF(y_0^*,.)$. That is, the correspondence from distance to default to TEDF is u-shaped.

Figure 3 plots TEDF of the dynamic firm together with the probability of paths that first hit the recapitalization trigger and default afterwards. It confirms that for well performing firms nearly the entire default probability comes from these recapitalization-paths.
The last two figures (Figure 4 and 5) shed light on the problem of underestimating TEDF when ignoring the firm’s opportunity to adjust its capital structure. Figure 4 shows that the underestimation is more severe for high-risk firms than for low-risk firms. Figure 5 illustrates that this effect is more pronounced for high-growth firms.

3.4 Value-at-Risk of Risky Debt

This subsection focuses on the Value-at-Risk (VaR) of a debt contract. In this context we examine the riskiness of a long-term investment into debt, where we understand long-term in the sense of an investment in a console bond that even

Figure 5: The underestimation of TEDFs when ignoring the opportunity to recapitalize plotted against the distance to default for different growth rates. The effect is more pronounced for high-growth firms.
outlasts possible reorganizations of the firm’s ownership or capital structure. That means, after bankruptcy as well as after debt is called the entire payoff is reinvested into console bonds of the firm. After such a reorganization the firm continues operation as an optimally levered firm, however, at a different scale (see Section 2).

Calculating the value at risk of a debt contract, the dynamics of the underlying $y_t$ (see Equation (3)) has to be translated into the dynamics of $D(y_t)$ in order to determine the respective quantile of the loss distribution. Furthermore, it has to be regarded that the outstanding principle (and thus the coupon flow) may change due to re-investment after bankruptcy or call of the existing bond (when call premium is positive, $\lambda > 0$). It is immediately evident that dynamic capital structure
adaptation affects the probability of events that cause restructuring of the firm, and consequently rescale the coupon. However, the entire shape of $D(y)$ changes when firms can call debt at some critical threshold $\gamma$ (see Figure 7), i.e., the dynamics of the value of the debt investment changes. A low call premium leads to a relatively flat value function and thus, the value of the bond is relatively insensitive to changes in $y$.

Figure 8 plots the typical shape of the one year 99% Value-at-Risk of an investment into corporate debt for dynamic and static debt level versus the distance to default. When the distance to default is low, the probability that the firm de-
Figure 8: The one-year Value-at-Risk at a 99% confidence interval of a debt contract for the base case parameters as function of the distance to default for static and dynamic debt level. It is implicitly assumed that the investor stays with the firm loyally even after reorganization, i.e., after debt is called as well as after bankruptcy the entire payoff is re-invested into bonds of the newly (and optimally) levered firm.

faults within the next year is greater than one percent, which means that the 99% VaR is determined by those scenarios, where the firm runs into bankruptcy and is restructured at a smaller scale but optimally levered. Since the investment is down-scaled in these cases (and hence the variance is reduced) and since the newly purchased bond of the optimally levered firm is less risky (see Figure 7, $D(y_0)$ is flatter than $D(y)$), bankruptcy defines something like a ‘catch tray’ for the value of the bond. That means as long as the probability of default exceeds one percent, the Value-at-Risk is simply defined by the loss that is given in the
case the firm runs into bankruptcy. Therefore, the Value-at-Risk increases for low DD (for both, the static as well as the dynamic debt structure firm). When DD is such that the probability of default is less that one percent (which is the case for \( DD \approx 2.4 \)) bankruptcy is not longer the dominant event and the VaR decreases. This is the case because \( D(y) \) becomes flatter for higher DD which reduces the sensitivity of the bond value to changes in \( y \). Since flattening is more pronounced in the case of dynamic capital structure, the reduction in VaR is greater for this firm. However, when the bond can be called at the critical restructuring level, the inverse leverage ratio jumps back to \( y_0 \), a level at which the downside risk is significantly increased. Therefore, for high DD the VaR is—for the callable bond—an increasing function, whereas the flattening of \( D \) persists the dominant factor reducing the VaR for large DD in the case of static debt level.

4 Conclusions

This paper has derived a structural credit risk model which explicitly accounts for the possibility that firms can alter their capital structure over time. The results show that the dynamics of firms’ financing decisions have important implications for the chosen leverage ratios and the required credit risk spreads.

We find that the option to adjust capital structure over time makes firms choose a lower initial leverage ratio. Despite this fact, bondholders require higher credit spreads to compensate them for the risk of future leverage increases.

The numerical analysis also produces estimates for the significance of model risk, i.e. the mistake that is made by using a static model. We find that, if a static model is used to infer the value of the firm’s assets and its volatility from
observed equity prices, then this can lead to substantial underestimation of fair credit spreads. The magnitude of the mistake increases with the volatility of the underlying asset value and with the tax benefit of debt and decreases with the premium that must be paid to old debtholders before a leverage increase.

We also analyze the relationship between the distance to default and expected default frequencies. We find that this relationship is non monotonic. While the expected default frequency initially decreases with the distance to default, it actually increases for high values of the distance to default. This happens since the probability of a leverage increasing capital structure adjustment increases with the firm’s distance to default. As a result the relationship between expected default frequency and distance to default is u-shaped. Compared to the results from the dynamic model, the static model significantly underestimates credit risk for large distance to default values. This result is consistent with the observation that the empirical default frequency decreases much slower than the theoretical relationship would imply.2.

An important implication of our numerical results is that the relationship between a firm’s distance to default and its expected default frequency crucially depends on firm characteristics. In particular the relationship depends on the volatility of the underlying cash flow process, the expected growth of the firm’s cash flow, and the costs of recapitalization, including the call premium. Thus, the analysis strongly suggests that one should condition on these characteristics when estimating the empirical relationship between the distance to default and expected default frequencies.

Several of our results could be tested empirically. First, expected default prob-

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2See, for example, the graph in Crouhy, Galai, and Mark (2000)
abilities are predicted to be non-monotonic in the firms’ distances to default. Second, the non-monotonicity should be particularly pronounced for high-risk firms and high-growth firms. Also, for firms with low effective corporate tax rates, i.e. firms with large non-debt tax shields the non-monotonicity should be found to be less significant.

References


