Business Valuation with Attention to 
Imputed Interest on Equity Increase –
A Comparison of Alternative Pricing Methods 
in a Model with Stochastic Profitability

Abstract
As early as the 1980s, several European countries implemented tax systems with imputed equity interest provisions. Since its tax reform in 2000, Austria has also allowed the deduction of (fictitious) imputed equity interest from the tax base. This paper integrates the resulting tax benefits related to equity into the valuation of corporations. Using the equity method with attention to the deductibility of imputed equity interest, the value of a business is calculated in a multi-period model. In this context, a market-to-book ratio endogenous to the model results for the equity of the valuated business. It is then demonstrated how to apply the APV method in a model with imputed equity interest. For each business - and thus also for an unlevered company - an adjustment is necessary to account for the tax shield resulting from equity financing. Therefore an unlevered company which does not use equity tax shields has to be chosen as a common denominator in the application of the APV method, in which a closed-form solution is presented for the value of this tax benefit. Intertemporal differences in risk make it necessary to use various risk-adjusted term structures of interest rates in the equity method as well as the APV method. Finally, the weighted average cost of capital in consideration of the deductibility of imputed equity interest is derived. A sensitivity analysis is conducted for all value components of a business and the weighted average cost of capital.

JEL classification: G24, H25, K34, M40

Keywords: Business Valuation, Adjusted Present Value, Imputed Equity Interest, Dual Income Tax System

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1 Introduction

Since the 1980s, new tax legislation in several countries has created an increasing number of tax systems in which the deductibility of imputed equity interest compensates (at least in part) for the preferential tax treatment of debt (e.g., the tax systems in Croatia, Finland, Denmark, Norway and Italy, as well as the proposed "Allowance for Corporate Equity" in the United Kingdom). All of these are dual income tax systems in which part of the business's earnings (i.e., the imputed equity interest) are taxed at a reduced rate.

Since its tax reform in 2000, Austrian legislation has also allowed (fictitious) imputed equity interest to be deducted from the tax base. In contrast to the tax systems listed in the previous paragraph, in which imputed equity interest is based on equity holdings, the Austrian variant calculates imputed equity interest on the basis of the annual increase in equity. From a fiscal standpoint, this implies less of a burden on the government's budget. These legal developments were observed both in Austria and abroad with great interest (see Wagner/Wenger (1999)).

Studies on the optimal capital structure for various tax systems with imputed equity interest can be found, for example, in Boadway/Bruce (1984), Devereux/Freeman (1991), Bogner/Frühwirth/Höger (1999), Arachi/Alworth (2000) and Bogner/Frühwirth/Höger (2001). For the Austrian tax system, Bogner/Frühwirth/Höger (1999) and Bogner/Frühwirth/Höger (2001) come to the conclusion that even with these new regulations, a maximum of debt financing is optimal for tax purposes.

In reality, however, equity financing is indeed used beyond the legal minimum. This is shown in the average annual growth in book equity for companies listed on the prime segment at the Vienna Stock Exchange from 1995 to 1999 (Table 2, Appendix). With only one exception, all of the companies examined recorded an increase in equity ranging from 4.85% p.a. to

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1 A description of the regulations in Scandinavian countries can be found, for example, in Soerensen (1998). The Croatian tax system is described in Rose/Wiswesser (1998). For Italy, see Bordignon/Giannini/Panteghini (2000) or Arachi/Alworth (2000). The proposal for the United Kingdom can be found in IFS (1991).

2 Those companies which were listed on the A Segment at the Vienna Stock Exchange as of April 10, 2001, were selected; the companies for which data was not available over the entire period were disregarded.
34.11% p.a. The average increase in equity for all companies included in the study was 12% p.a. These growth rates relate to an environment without equity financing benefiting from favorable tax provisions, i.e. the imputed equity interest provisions in Austria introduces as of 2000. With these provisions in place it seems plausible to see even higher growth rates in the future.

This high level of equity financing despite its tax disadvantages might be attributed to non-tax-related arguments (e.g., agency costs of debt). In any case, it makes it necessary to integrate the deductions allowed by the law for imputed interest on increased equity into existing business valuation models. This step is especially relevant for growth businesses which use equity to finance at least part of their growth. While the literature on capital structure optimization in light of imputed equity interest does provide solutions, these are yet to be developed in the field of business valuation in consideration of imputed equity interest.

The main objective of this paper is to fill this gap.

In business valuation, Discounted Cash Flow methods dominate the field. These methods include the equity method, the APV method and the entity method (see Copeland/Koller/Murrin (2000)).

Under the equity method, payments to shareholders (after all business taxes) are discounted using their cost of capital. In the APV method, the value of the levered company is found by calculating the value of an equivalent unlevered company and correcting this figure to account for the effects of financing decisions on value, particularly the tax benefit arising from debt (e.g., see Brealey/Myers (2000)). Under the entity method, payments to all investors (after all business taxes) are discounted using the weighted average cost of capital (WACC).

In this paper, the equity method, APV method and entity method are extended to include the deductibility of imputed equity interest in a multi-period value-driver model under risk. The results provide valuation methods for corporations which take advantage of their ability to deduct imputed interest on increased equity. In the process, a market-to-book ratio endogenous to the model arises for the valuated business. In addition, this paper extends the APV method to an environment where imputed equity interest can be deducted from the tax base. An adjustment to account for the tax shield arising from imputed interest on increased equity is necessary both for the levered and unlevered companies. We also show that an unlevered company which does not use equity tax shields has to be chosen as a common denominator in the application of the APV method. A closed-form solution is derived for the value of the tax benefit arising from equity financing. Intertemporary differences in risk make it necessary to use various risk-adjusted term structures of interest rates in the equity method.
as well as the APV method. It is further demonstrated that the interest rates for a business financed solely by equity (without using equity tax shields) are to be used to discount the tax shields from imputed equity interest. Under the entity method, a closed-form solution for the weighted average cost of capital with attention to imputed equity interest is derived. A sensitivity analysis is carried out for all value components of a business and for the weighted average cost of capital.

This paper is structured as follows: Chapter 2 gives an overview of the Austrian tax system after the tax reforms of 2000. Chapter 3 presents the structure of the model. Chapter 4 uses the equity method to calculate the business value in consideration of imputed interest on increased equity, and Chapter 5 integrates imputed interest on increased equity into the APV method. In Chapter 6, a sensitivity analysis is performed, and in Chapter 7 the entity method is analyzed with regard to imputed interest on increased equity.

2 Overview of the Austrian Tax System after the 2000 Tax Reforms

First, the regulations applying to corporations before the tax reforms of 2000 are described: Corporate earnings are subject to a corporate tax of 34%. Debt interest paid by the business can be deducted from the corporate tax base.

Those additional regulations in the Austrian Corporate Tax Act (KöStG) with regard to imputed interest on increased equity which are required for the specific purposes of this paper are described below.

Interest on the increase in book equity recorded in a business year is to be calculated in the following way: The applicable interest rate is the average of secondary market yields for all issuers on the domestic bond market from January to December, increased by 0.8. This interest rate is applied to the equity increase over the corresponding year.

The resulting interest on the equity increase can be deducted as an operating expense. The amount deducted is to be recorded under “special earnings”, which are to be taxed at a rate of 25%.

Therefore, this is in effect a dual income tax system in which part of the earnings - the imputed normal return to the increase in the equity - are taxed at the reduced rate of 25%, while the remaining profits are taxed at the ordinary corporate tax rate of 34%.
3 Model

This analysis is based on a capital market as described in Fama (1977). In addition, it is assumed that the risk-free interest rate as well as the market price of risk are constant. The risk-free interest rate is denoted by $r_f$, and the market price of risk is represented by $\lambda$.

The subject of the study is a corporation with an infinite life span, which has riskless debt outstanding in the form of perpetual bonds with a par value of $D_t$ at any time $t$. The capital structure at book values, denoted by $\nu = D_t/EQ_t$, is assumed to be constant over time.

The model involves uncertainty. This uncertainty arises from the stochastic development of the book return on investment before interest and taxes in each year $t$ (symbolized by $ROI_t$), which is calculated as the ratio of the earnings before interest and taxes in year $t$, $EBIT_t$, to the total book capital at the beginning of the year $t$ after the distribution of earnings from the year $t-1$, $TC_{t-1}$ ($=$ total capital at the end of year $t-1$, specifically: after payment of business taxes, distribution of earnings and capital increase for the year $t-1$). This model is based on the book return on investment because the imputed interest on increased equity is based on book equity (see Chapter 2). The book returns on investment for each year are uncorrelated. The return on investment in year $t$ expected on the basis of the information available at $t-1$ is constant over time and denoted by $\overline{ROI}$. It is assumed that in any case the return on investment is sufficiently high to allow the use of all tax shields. The covariance (on the basis of the information available as of $t-1$) between the return on investment over year $t$ and the return of the market portfolio over year $t$ is constant over time and symbolized by $\rho$.

The tax system is modelled as follows: Corporate earnings are subject to corporate tax at the rate $\tau_K$. Debt interest and (fictitious) imputed interest on increased equity can be deducted from earnings for corporate tax purposes. The imputed interest on increased equity is calculated by multiplying the increase in book equity by the equity interest rate $re$. Taxes for the year $t$ are to be paid at the end of the year $t$. The increase in equity each year is the difference between equity at the end of the year (after the deduction of taxes, the distribution

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3 As a generalization of this model, the assumption of a 2- or multi-stage model would be possible. In order to implement this generalization, the expected development of the return on investment would have to be predicted for each stage. The model presented in this paper would correspond to the final stage of such a multi-stage model and is thus a prerequisite for the implementation of a multi-stage model. The methodology to be applied to the valuation of payments in the previous stages is completely identical to that in the final stage.

4 Under a constant book-to-market ratio, this is identical to the frequent assumption of a constant capital structure at market values (e.g., see Ross/Westerfield/Jaffee (1999) or Miles/Ezzel (1980)). In this paper, a constant capital structure at book values is assumed, as imputed equity interest is calculated on the basis of book equity.

5 In principle, it would be possible to deviate from this assumption; however, this would generate option-like profiles for the tax benefits arising from equity as well as debt.
of the dividend for that year and the capital increase for the year, which is described below) and the equity at the end of the previous year (likewise after the deduction of taxes, the distribution of the dividend for that year and the capital increase for the year).  

The equity interest rate $r_e$ is assumed to be constant over time. Deductions for imputed interest on increased equity are subject to a special earnings tax at the rate $\tau$. As is common in business valuation, personal taxes are disregarded in this paper (e.g., see Ross/Westerfield/Jaffee (2001)).

The earnings after taxes for each year $t$ are paid out in full at the end of the year $t$ (full payout hypothesis in business valuation). At the end of the year $t$, the total capital is increased by the product of EBIT in year $t$, $EBIT_t$, and the factor $q$. This manner of modelling implies that more capital flows into businesses which are (expected to be) more profitable than into businesses which are (expected to be) less profitable. Furthermore, in states of higher earnings, more capital is injected into the business than in states with lower earnings. Both effects appear quite plausible, especially when positively auto-correlated earnings are assumed. The parameter $q$ is a consequence of the investment program and of the business's resulting financing needs. This parameter can be interpreted intuitively as the plowback ratio.  

The increase of total capital by means of equity or debt is performed with due attention to the constant capital structure, $\nu = D_t/EQ_t$.

In conclusion, the following flows of payments thus take place at the end of the year: First earnings before interest and taxes are realized, and (depending on EBIT) an increase in capital is performed, from which the imputed interest on increased equity - and thus the tax liability - results. Finally, earnings after taxes are distributed in full.

4 Equity Method

The equity method uses dividend payments to equity holders, in consideration of imputed interest on increased equity and debt interest (under the full payout hypothesis, this is equal to earnings after taxes, $EAT_t$) minus equity increases $CI_t$; in total this equals the free cash flow, $F"

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6 This manner of modelling increased equity corresponds to its legal definition under the Austrian Corporate Tax Act as long as it is assumed that all revenues and expenses are incurred on January 1st and that all dividends, capital increases and business taxes are paid on January 1st. Otherwise, the average equity over the year calculated on a daily basis would have to be applied. However, deviating from this assumption would increase the complexity of the model unnecessarily, without generating any additional results.

7 In this context, the plowback ratio is based on pre-tax earnings. Modelling plowback as a function of after-tax earnings would result in varied growth according to each business's capital structure, and it would then no longer be possible to separate investment and financing decisions. In particular, this would also contradict the assumption of a perfect capital market.
\[ FCF_t = EAT_t - CI_t \]. These free cash flows have to be discounted using an interest rate adjusted to account for imputed interest on increased equity and financial risk.

### 4.1 Growth in Expected Equity

The total book capital, equity and debt capital (each after capital increase, earnings distribution and tax payments for year \( t \)) are denoted below as \( TC_t, EQ_t \) and \( Dt \), respectively. Our first step is to show that expected equity increases at the constant, positive growth rate \( qROI \):

Because earnings after taxes are distributed in full, the total capital at the end of year \( t \) is the sum of the total capital at the beginning of the year, \( TC_{t-1} \), and the increase in the total capital:

\[
TC_t = TC_{t-1} + qEBIT_t = TC_{t-1} + qTC_{t-1}ROI_t = TC_{t-1}(1 + qROI_t)
\]

Because the increase in the total book capital \( qEBIT_t = qTC_{t-1}ROI_t \) and the book capital structure remains constant, the relative capital increase must be the same for equity and debt. The increase in equity is thus \( CI_t = qEQ_{t-1}ROI_t \).

Therefore,

\[
EQ_t = EQ_{t-1} + qEQ_{t-1}ROI_t = EQ_{t-1}(1 + qROI_t)
\]

The expected value of \( EQ_t \), conditional on the information available as of \( t-1 \), is

\[
E_{t-1}[EQ_t] = E_{t-1}[EQ_{t-1}(1 + qROI_t)] = EQ_{t-1}(1 + qROI)
\]

where \( E[\bullet] \) denotes the expected value conditional on the information available at \( t \). The unconditional expected value of equity using the law of iterative expectations:

\[
E[EQ_t] = EQ_t(1 + \overline{ROI})
\]

It has therefore been shown the expected equity increases at a rate of \( \overline{ROI} \).

### 4.2 Free Cash Flows

In the next step, the free cash flow to shareholders, which is necessary for valuation, is derived. In year \( t \), EBIT equals \( TC_{t-1}ROI_t \). Because debt is risk-free, debt interest amounts to \( Dt_{t-1}r_f \). In year \( t \), EBT is thus:

\[
EBT_t = TC_{t-1}ROI_t - Dt_{t-1}r_f = EQ_{t-1}ROI_t + EQ_{t-1}v(ROI_t - r_f) = EQ_{t-1}(ROI_t + vROI_t - vr_f)
\]

This amount is split up into taxes and earnings after taxes, the latter being distributed in full to shareholders.
As described in the previous section, the increase in equity is $EQ_{t-1}qROI_t$. Thus the imputed interest on increased equity in year $t$ amounts to:

$$IIE_t = EQ_{t-1}qROI_t r_e$$

This amount forms the basis for the special earnings tax for the year $t$. The corporate tax base in year $t$, $TB_t$, is the difference between $EBT_t$ and $IIE_t$:

$$TB_t = EQ_{t-1}(ROI_t + vROI_t - vr_f) - EQ_{t-1}qROI_t r_e = EQ_{t-1}(ROI_t + vROI_t - vr_f - qROI_t r_e)$$

The tax liability (corporate tax and special earnings tax) in year $t$, $TAX_t$, is thus:

$$TAX_t = EQ_{t-1}(ROI_t + vROI_t - vr_f - qROI_t r_e)\tau_K + EQ_{t-1}qROI_t r_e \tau_S =$$

$$= EQ_{t-1}[[(ROI_t + vROI_t - vr_f)\tau_K - qROI_t r_e(\tau_K - \tau_S)]]$$

The expression ($\tau_K - \tau_S$) represents the differential between the reduced tax rate and the ordinary tax rate in the dual income tax system. The after-tax earnings in year $t$, denoted by $EAT_t$, is the difference between earnings before taxes, $EBT_t$, and the tax liability in year $t$, $TAX_t$:

$$EAT_t = EQ_{t-1}(ROI_t + vROI_t - vr_f) -$$

$$= EQ_{t-1}[[(ROI_t + vROI_t - vr_f)(\tau_K - \tau_S)]]$$

These earnings after taxes are distributed in full to shareholders. In turn, the amount $qEQ_{t-1}ROI_t$ is returned to the business in the form of increases in equity. The free cash flow to shareholders is thus:

$$FCF_t = EQ_{t-1}[[(ROI_t + vROI_t - vr_f)(1 - \tau_K) + qROI_t r_e(\tau_K - \tau_S)] - EQ_{t-1}qROI_t] =$$

$$= EQ_{t-1}[ROI_t[(1 + v)(1 - \tau_K) + q\tau_K r_e(\tau_K - \tau_S)] - vr_f(1 - \tau_K)]$$

Thus the free cash flow of year $t$ is composed of the after tax earnings of an equivalent unlevered company which does not use equity tax shields, $EQ_{t-1}ROI_t(1 + v)(1 - \tau_K)$, a reduction for the increase in equity $EQ_{t-1}ROI_t q$, the equity tax shields $EQ_{t-1}ROI_t q\tau_K r_e(\tau_K - \tau_S)$ and the interest charge reduced by the resulting debt tax shields, $EQ_{t-1}vr_f(1 - \tau_K)$.

### 4.3 Valuation using the Equity Method

First, the present value of the free cash flow to shareholders in year $t$ is calculated:
\[ PV(FCF_t) = \frac{E[FCF_t]}{\prod_{s=1}^t [1+k_E(s,t)]} \]

where \( k_E(s,t) \) is the interest rate for discounting the free cash flow to shareholders at \( t \), \( FCF_t \), for the period from \( s \) to \( s-1 \). In the terminology of term structure theory, this is a one-period forward rate.

Because the return on investment is uncorrelated in different periods, the (unconditional) expected free cash flow is:

\[ E[FCF_t] = EQ_0 \left( 1 + \frac{\overline{ROI} \cdot K_f - S - K_e}{\overline{ROI} \cdot K_f - S - K_e} \right) - q \cdot \left( 1 - \tau_K \right) \]

**Proposition 1:** The interest rate \( k_E(s,t) \) is:

\[
k_E(s,t) = \left( 1 + r_f \right) \frac{\overline{ROI} \cdot [1 + \frac{\overline{ROI} \cdot (1 + \tau_K) + qr_s (\tau_K - \tau_s) - q]}{-qr_s (\tau_K - \tau_s) - q} - vr_f (1 - \tau_K) - 1 \quad \text{where } s = t
\]

\[
k_E(s,t) = \left( 1 + r_f \right) \frac{1 + q\overline{ROI}}{1 + q\overline{ROI} - \lambda \rho} - 1 \quad \text{where } s < t
\]

These interest rates are derived in the Appendix.

The expression \( \overline{ROI} - \lambda \rho \), henceforth denoted as \( \overline{ROI} \), can be interpreted as the certainty equivalent of the return on investment and thus as the risk-adjusted expected return on investment. The expression \( q\overline{ROI} \) is the certainty equivalent of the relative growth and thus the risk-adjusted expected relative growth. Therefore, the risk-adjusted interest rate where \( s < t \) is calculated using the risk-free interest rate and the ratio of the actual growth factor to the risk-adjusted growth factor.

The difference in \( k_E(s,t) \) between the cases \( s = t \) and \( s < t \) can be explained as follows: In cases where \( s < t \), the systematic risk of the ratio between two conditional expected values of \( FCF_t \) (conditional on the information available at different points in time) is relevant, while in the case where \( s = t \) the systematic risk of the ratio of the actual payment \( FCF_t \) to the expected value (conditional on the information available as of \( t-1 \)) of \( FCF_t \) is relevant, which leads to a corresponding increase in complexity in the interest rate when \( s = t \) (see Appendix).

Thus the value of free cash flow in year \( t \) is:

\[
PV(FCF_t) = \frac{EQ_0 \left( 1 + \frac{\overline{ROI} \cdot [1 + \frac{\overline{ROI} \cdot (1 + \tau_K) + qr_s (\tau_K - \tau_s) - q]}{-qr_s (\tau_K - \tau_s) - q} - vr_f (1 - \tau_K)}{\overline{ROI} \cdot [1 + \frac{\overline{ROI} \cdot (1 + \tau_K) + qr_s (\tau_K - \tau_s) - q]}{-qr_s (\tau_K - \tau_s) - q} - vr_f (1 - \tau_K)} - 1 \right)}{\left( 1 + r_f \right) \frac{1 + q\overline{ROI}}{1 + q\overline{ROI} - \lambda \rho} - 1} \]

The numerator in the expression above is the certainty equivalent of the free cash flow in year \( t \).

Now that the value of the free cash flow in year \( t \) has been found, the value of all free cash flows to equity holders is to be determined in order to find the value of the company’s equity.

**Proposition 2:** Assuming an infinite life span, the value of equity is:

\[
V_{EQ} = \frac{EQ_0 [1 + qROI]}{r_f - qROI} = \frac{EQ_0 \bar{ROI} [(1 + v)(1 - \tau_K) + qr_s(\tau_K - \tau_S) - q] - vr_f (1 - \tau_K)}{1 + r_f}
\]

\( V_{EQ} \) is derived in the Appendix.

Here it becomes clear that the expected return on investment \( \bar{ROI} \), market price of risk \( \lambda \) and covariance \( \rho \) variables are only contained in \( \bar{ROI} = \bar{ROI} - \lambda \rho \). The return on investment is thus only accounted for in its risk-adjusted form. The value of equity naturally increases along with the risk-adjusted expected return on investment. Thus the lower the systematic risk in the return on investment becomes, the higher the value of equity is.

The value of equity can thus be explained more simply by applying the present value formula for a constantly growing perpetuity to the aforementioned certainty equivalents. The numerator of \( V_{EQ} \) corresponds to the certainty equivalent of the first free cash flow. In the denominator, \( qROI \) denotes the risk-adjusted expected relative growth in the certainty equivalents. Because risk adjustment was already performed for the free cash flows by using certainty equivalents, \( r_f \) is to be used as the interest rate.

From the equation for the value of equity, an endogenous market-to-book ratio - \( MBR \) - for the equity of the valued business arises:

\[
MBR = \frac{V_{EQ}}{EQ_0} = \frac{\bar{ROI} [(1 + v)(1 - \tau_K) + qr_s(\tau_K - \tau_S) - q] - vr_f (1 - \tau_K)}{r_f - qROI}
\]

The determinants and sensitivities (with the exception of factor \( EQ_0 \)) correspond to those in \( V_{EQ} \). An analysis of these sensitivities will follow in Chapter 6.

The total value of the business, which is denoted by \( V \), is then calculated by adding the values of equity and debt, each at time 0.

\[
V = V_{EQ} + D_0
\]
5 APV Method

In its basic form, the APV method is based on the work of Myers (1974). The source of this approach lies in a model world in which there is only one business tax with deductible debt interest. The value of the business is split up into the "value of the unlevered company" and the "value of tax shields from debt financing". Brealey/Myers (2000) define "adjusted present value" on p. 1061 in more general terms: the "net present value of an asset if financed solely by equity plus the present value of any financing side effects".

Accordingly, the APV approach is also applied to real and far more complex tax systems. For example, Hachmeister (1999) proves that the APV approach is appropriate for the German tax system, and Drukarczyk/Richter (1995) use the APV approach for the valuation of financing effects rooted in unique characteristics of the German tax system. Monkhouse (1997) also uses the APV approach for business valuation within the Australian tax system.

The objective of this chapter is to extend the APV approach to a tax system which has been broadened to include imputed equity interest. If the APV approach is applied to such a tax system, the following equation results:

\[ V = V^U + PV(TSD) + PV(TSE) \]

where

- \( V \) = Market value of the business after corporate tax with imputed interest on increased equity
- \( V^U \) = Market value (of equity) of a company which is unlevered but otherwise equivalent (especially in terms of growth and distribution policies), after corporate tax but before taking into account the tax benefit from imputed interest on increased equity
- \( PV(TSD) \) = Value of the tax shield arising from debt financing
- \( PV(TSE) \) = Value of the tax shield arising from imputed interest on increased equity for the levered company.

Thus a fictitious unlevered company which does not use equity tax shields has been selected as a common denominator. The reason behind this choice lies in the fact that even businesses financed solely by equity are not homogenous, as this group includes businesses with different growth rates and thus different equity tax shields.

The following remarks must be made regarding the two tax shields: Even in a world with imputed interest on increased equity, debt financing still offers an advantage - \( PV(TSD) \) - over equity financing. However, the tax benefit arising from equity financing, \( PV(TSE) \), has now been added to the existing tax benefit arising from debt financing. Due to the regulations
governing imputed interest on increased equity, an adjustment to account for this imputed interest is also necessary in the case of unlevered companies.

The goal of this chapter is the separate valuation of the three components $V^U$, $PV(TSD)$ and $PV(TSE)$. In order to verify whether the APV method is applicable in a world with imputed equity interest, the equivalence between the value found using the equity method and the value found under the APV method is then examined.

5.1 Valuation of the Unlevered Company without using Equity Tax Shields

In contrast to the levered company, the unlevered (but otherwise equivalent) company has equity which is equal to the total capital of the levered business at time 0:

$$EQ_0^U = TC_0 = EQ_0(1+\nu)$$

For the purpose of valuation, it is first necessary to examine the free cash flow of the unlevered company for any year $t$, denoted by $FCF^U_t$. The value of this free cash flow is:

$$PV(FCF^U_t) = \frac{E[FCF^U_t]}{\prod_{s=1}^t [1 + k_U(s,t)]}$$

where $k_U(s,t)$ stands for the risk-adjusted interest rate (for the period from $s-1$ to $s$) for valuating the unlevered business’ free cash flow in year $t$ before consideration of the tax shield arising from imputed equity interest.

The free cash flow of the unlevered company is calculated as follows: EBIT for the unlevered company in year $t$, which is equal to EBT in the same year, is

$$EBIT^U_t = EQ^U_{t-1}ROI_t.$$  

Once corporate tax is subtracted, the remaining amount is $EAT^U_t = EQ^U_{t-1}ROI_t(1-\tau)$, which is distributed in full. In order to find the free cash flow, the funds flowing back into the business in the form of capital increases, $qEBIT^U_t = qEQ^U_{t-1}ROI_t$, have to be subtracted:

$$FCF^U_t = EQ^U_{t-1}ROI_t(1-\tau) - qEQ^U_{t-1}ROI_t = EQ^U_{t-1}ROI_t(1-\tau - q)$$

The expected value of the free cash flow in year $t$ is thus:

$$E[FCF^U_t] = E[EQ^U_{t-1}ROI_t(1-\tau - q)] = EQ^U_0 \left(1 + \frac{q \overline{ROI}}{\overline{ROI}}\right)^{t-1} \overline{ROI}(1-\tau - q)$$

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The growth rate of $E[EQ^U]$ equals the one of $E[EQ]$. The proof follows in analogy to Section 4.1.
Let us now turn to the interest rates $k_{U}(s,t)$:

**Proposition 3:**

a) When $s < t$, the interest rate $k_{U}(s,t)$ is identical to the one under the equity method, $k_{E}(s,t)$.

b) When $s = t$, the interest rate is

$$k_{U}(s,t) = \left(1 + r_{f}\right)\frac{\text{ROI}}{\text{ROI}} - 1$$

These interest rates are derived in the Appendix.

The ratio $\frac{\text{ROI}}{\text{ROI}}$ reflects the difference between the actual expected return on investment and the risk-adjusted expected return on investment. As $\frac{\text{ROI}}{\text{ROI}} = \frac{\text{ROI}}{\text{ROI}} - \lambda\rho$, the interest rate $k_{U}(s,t)$ rises along with the market price of risk and along with the covariance $\rho$.

The difference in $k_{E}(s,t)$ between the cases $s = t$ and $s < t$ can again be explained by the differences in systematic risk. In cases where $s < t$, the systematic risk is determined by the ratio between two conditional expected values of $FCF_{t}^{U}$ (conditional on the information available at different points in time), while in the case where $s = t$ the systematic risk of the ratio of the actual payment $FCF_{t}^{U}$ to the expected value (conditional on the information available as of $t-1$) of $FCF_{t}^{U}$ is relevant.

The connection between the interest rates $k_{U}(s,t)$ and $k_{E}(s,t)$ for cases where $s < t$ or $s = t$ can be explained as follows:

The difference between the levered and the unlevered company consists on the one hand in the amount of equity [i.e., $EQ_{0}^{U} = EQ_{0}(1+\nu)$], while on the other hand the unlevered company is subject to different tax treatment. This arises from the lack of debt tax shields as well as our definition of the unlevered company under the APV method (non-use of equity tax shields).

The difference in equity levels is irrelevant because the equity level is dropped in the derivation of the interest rate, so that this level has no influence on systematic risk.

The difference in tax treatment (elimination of tax shields for the unlevered company) only comes into play in the final year ($s = t$). This can be attributed to the fact that the corresponding differences in the ratio between two conditional expected values ($s < t$) cancel each other out, which is not the case in the ratio of the actual payment $FCF_{t}^{U}$ to the expected value conditional on the information available at $t-1$ ($s = t$).

In summary, the systematic risk in the free cash flow of the unlevered company is identical to that in the free cash flow of the levered business (under the equity method) when $s < t$. When $s = t$, the systematic risk in the free cash flow of the unlevered company is completely identical to the systematic risk in the return on investment.
The value of the free cash flow in year $t$ is thus:

$$
PV(FCF_t^U) = \frac{EQ_0^U \left(1 + q\overline{ROI}\right)^{-1} \overline{ROI}(1 - \tau_K - q)}{\left[\left(1 + r_f\right)\overline{ROI}\right] \left(1 + r_f\right)^{1 + q\overline{ROI}} 1 + q\overline{ROI}} = \frac{EQ_0^U \left(1 + q\overline{ROI}\right)^{-1} \overline{ROI}(1 - \tau_K - q)}{(1 + r_f)}
$$

The numerator in this expression is the certainty equivalent of the free cash flow for the unlevered company.

**Proposition 4:** Assuming an infinite life span, the value of the unlevered company is as follows:

$$
V^U = \frac{EQ_0^U \overline{ROI}(1 - \tau_K - q)}{r_f - q\overline{ROI}}
$$

A complete derivation is given in the Appendix. The economic interpretation corresponds to its counterpart under the equity method. The numerator contains the certainty equivalent of the first free cash flow, $q\overline{ROI}$ is the risk-adjusted expected growth rate, and $r_f$ results from the use of the certainty equivalents.

### 5.2 Valuation of Debt Tax Shields

First the debt tax shields in year $t$ are examined. The present value of these tax shields is:

$$
PV(TSD_t) = \frac{E[TSD_t]}{\prod_{s=1}^t [1 + k_{TSD}(s, t)]}
$$

where $k_{TSD}(s, t)$ is the risk-adjusted interest rate (for the period from $s-1$ to $s$) for valuating the debt tax shields in year $t$.

The debt tax shields in year $t$ are $TSD_t = D_t r_f \tau_K = EQ_{t, i} vr_f \tau_K$. The expected payment thus amounts to:

$$
E[TSD_t] = EQ_0^U \left(1 + q\overline{ROI}\right)^{-1} vr_f \tau_K.
$$
The following proposition applies to interest rates:

**Proposition 5:**

a) In cases where \( s < t \), the interest rate \( k_{TSD}(s,t) \) is identical to its counterpart under the equity method, \( k_E(s,t) \).

b) In the case where \( s = t \), the interest rate is the risk-free interest rate \( r_f \).

A complete proof is provided in the Appendix.

The economic interpretation in the cases where \( s < t \) is as follows: Debt is indeed risk-free, but the amount of (the tax benefit arising from) this debt is not. Due to the constant capital structure, the systematic risk is equivalent to that of equity under the equity method - and in cases where \( \rho > 0 \), \( k_E(s,t) \) is greater than the risk-free interest rate.

The case where \( s = t \) can be interpreted as follows: Because debt interest is predictable (for one period), debt interest at the end of year \( t \) - and thus the amount of the tax benefit arising from this debt interest - is already known at the beginning of year \( t \). Thus the systematic risk in debt tax shields is 0 in the final year \( t \) (for more on both cases, see also Miles/Ezzel (1980)).

On the basis of these interest rates, the value of the debt tax shields in year \( t \) can be found:

\[
P V (T S D) = \frac{E Q_0 \left(1 + q R O I \right)^{-1} v r_f \tau_k}{\left(1 + r_f \right) \left[1 + r_f \left(1 + q R O I \right) \right]^{n-1}} \frac{E Q_0 \left(1 + q R O I \right)^{-1} v r_f \tau_k}{\left(1 + r_f \right) \left[1 + r_f \left(1 + q R O I \right) \right]^{n-1}}
\]

The numerator in the final expression corresponds to the certainty equivalent of the debt tax shields in year \( t \).

**Proposition 6:** For a business with an infinite life span, the value of all debt tax shields is:

\[
P V (T S D) = \frac{E Q_0 v r_f \tau_k}{r_f - q R O I}
\]

The derivation is given in the Appendix.

The numerator contains the debt tax shields for the first period. A risk adjustment is not performed for these tax shields because according to our model (see Chapter 3) these can be used in any case and \( E Q_0 \) is known at time 0, thus making the tax shields certain in the first year. Due to the assumption of a constant capital structure, the tax shields in the ensuing years are subject to the same risk as equity is (with a delay of one year). For this reason, the growth rate to be used is once again the risk-adjusted growth rate \( q R O I \).
5.3 Valuation of Equity Tax Shields

The valuation of equity tax shields is performed using the known pattern:

\[ PV(TSE_t) = \frac{E[TSE_t]}{\prod_{s=1}^{t-1} [1 + k_{TSE}(s,t)]} \]

where \( k_{TSE}(s,t) \) is the risk-adjusted interest rate (for the period from \( s-1 \) to \( s \)) for valuating equity tax shields in year \( t \).

The imputed interest on increased equity in year \( t \) is mentioned in Chapter 4:

\[ II_{I_t} = EQ_{t-1,ROI,q_r} \]

If the effects of corporate tax and the special earnings tax are taken into account, the net tax benefit from imputed interest on increased equity in year \( t \) is:

\[ TSE_t = EQ_{t-1,ROI,q_r}(\tau_k - \tau_s) \]

The expected value in the numerator is thus:

\[ E[TSE_t] = EQ_0 (1 + \bar{ROI})^{-1} \frac{ROI}{ROI} q_r (\tau_k - \tau_s) \]

**Proposition 7:** For all periods (\( s < t \) and \( s = t \)), the interest rates for the tax shield arising from imputed equity interest, \( k_{TSE}(s,t) \), are identical to those used for the unlevered company which does not use equity tax shields, \( k_{U}(s,t) \).

A proof is provided in the Appendix.

The reason for the equality of \( k_{TSE}(s,t) \) and \( k_{U}(s,t) \) lies in the risk equivalence with regard to the market risk of equity tax shields and free cash flows for the unlevered business.

In cases where \( s < t \), \( k_{TSE}(s,t) \) is equal to the interest rate for debt tax shields. This is again justified by the fact that both types of tax shields are subject to the same systematic risk, that is, the equity risk. In contrast to equity tax shields, however, the debt tax shields are risk-free in the last year \( s=t \), as described above.

Using the interest rates derived, the value of the tax benefit arising from imputed equity interest in year \( t \) is found as follows:

\[ PV(TSE_t) = EQ_0 (1 + \bar{ROI})^{-1} \frac{ROI}{ROI} q_r (\tau_k - \tau_s) = \]

\[ = EQ_0 (1 + \bar{ROI})^{-1} \frac{ROI}{ROI} q_r (\tau_k - \tau_s) \]

\[ = EQ_0 (1 + \bar{ROI})^{-1} \frac{ROI}{ROI} q_r (\tau_k - \tau_s) \]
Again, the numerator in the second expression is the corresponding certainty equivalent.

**Proposition 8:** On the basis of the infinite life span, the value of the equity tax shields is as follows:

\[ PV(TSE) = \frac{EQ_{0, \bar{ROI}}q_{r}(\tau_{K} - \tau_{S})}{r_{f} - qROI} \]

A derivation is given in the Appendix.

The numerator contains the equity tax shields from the first year. In contrast to the debt tax shields in the first year, a risk adjustment is necessary here because the increase in equity over the first year is subject to the same risk as the return on investment in the first year. Equity tax shields grow at the risk-adjusted expected growth rate \( qROI \) in the ensuing years, as is the case under the equity method.

### 5.4 APV Method Summary

The total value of the unlevered business which does not use equity tax shields and the value of debt and equity tax shields, is the value of the business using the APV method.

**Proposition 9:** The value of the business under the APV method is the same as the value of the business under the equity method.

A complete proof for Proposition 9 is provided in the Appendix.

### 6 Sensitivity Analysis

In this chapter an analytical sensitivity analysis is performed. The entire derivation (especially of the first derivatives involved) is available from the authors upon request. In this chapter we have restricted our analysis to the economic interpretation.

The formulas derived in the preceding chapters for the value of the levered/unlevered company and for the value of debt and equity tax shields, require the following restrictions:

\[ 1 - \tau_{K} > \frac{r_{f}}{\bar{ROI}} > q \]

These restrictions imply that the risk-adjusted return on investment after corporate tax, without allowing for tax shields, exceeds the risk-free rate and that the risk-adjusted expected growth rate is lower than the risk-free rate. The first inequality holds for plausibility reasons,
and the second inequality guarantees finite business values. Based on these restrictions, we can identify the following influence of each factor on the respective value components:

<table>
<thead>
<tr>
<th>Factor/Value</th>
<th>( V_{EQ} )</th>
<th>( V )</th>
<th>( V^U )</th>
<th>( PV(TSD) )</th>
<th>( PV(TSE) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r_f )</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>( \lambda )</td>
<td>-/+</td>
<td>-/+</td>
<td>-/+</td>
<td>-/+</td>
<td>-/+</td>
</tr>
<tr>
<td>( \rho )</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>ROI</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>( q )</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>=</td>
</tr>
<tr>
<td>( r_e )</td>
<td>+</td>
<td>+</td>
<td>=</td>
<td>=</td>
<td>+</td>
</tr>
</tbody>
</table>

Table 1: Sensitivity analysis

A positive (negative) sign indicates the positive (negative) influence of the variable in each row on the value component represented by each column. An equal sign shows that the variable in question has no influence on the value component analyzed. All of these sensitivities demonstrate the plausibility of the model.

A higher risk-free interest rate logically leads to deeper discounting of all payments. However, an increase in \( r_f \) also causes an increase in debt tax shields - an effect which is overcompensated by the former effect in all value components, even in \( PV(TSD) \), whereby all of the value components analyzed decrease. Because risk adjustment is necessary in all valuation formulae, an increase in the market price of risk \( \lambda \) causes decreased/increased values depending on the sign of \( \rho \). For a negative correlation \( \rho \) risk adjustment results in an increase of the corresponding certainty equivalents. For the more common case of a positive correlation \( \rho \), certainty equivalents decrease with a higher market price of risk \( \lambda \). An analogous argument applies to the covariance between the market return and the return on investment, as increased covariance implies increased systematic risk. Increasing the expected return on investment raises all payments due to higher business growth, which has the effect of increasing value if risk remains the same (c. p.). The same applies to increasing the plowback ratio \( q \). An increase in the business’s debt \( v \) implies a larger business; because profitability (c. p.) remains the same, this likewise increases value. Only in the case of tax

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9 This analysis is based on the fact that \( \tau_k > \tau_s \) which guarantees positive equity tax shields.

10 The effect of \( \lambda \) on the valuation formulae is divided into two components. On the one hand \( \lambda \) influences the certainty equivalent of the free cash flows, on the other hand \( \lambda \) affects the risk adjusted expected growth rate. Both components have the same direction, depending on the covariance parameter \( \rho \).
shield arising from imputed interest on increased equity does debt have no effect, as only the equity component is analyzed. An increase in the equity interest rate does not affect the unlevered company (without imputed equity interest), nor the value of debt tax shields, while positive effects can be seen for all other value components. This positive effect arises from the increase in the equity tax shields.

Interesting insights (not shown in Table 1) arise when a constant spread between \( r_e \) and \( r_f \) is assumed, as specified in Austrian tax legislation. In this model reparameterization, an increase in \( r_f \) (and thus also \( r_e \)) leads to a decrease in all of the five values. This can be explained by the interaction of the following effects: On the one hand, an increase in \( r_f \) leads to a deeper discounting of all payments. On the other hand, the accompanying increase in \( r_e \) causes an increase in the (expected) equity tax shields in each year - an effect which, however, is overridden by the former effect in all value components, even in \( PV(TSE) \).

7 Entity Method

The objective of this chapter is to determine the weighted average cost of capital (WACC) in consideration of imputed interest on increased equity. The WACC, \( w \), is implicitly defined as follows:

\[
\lim_{T \to \infty} \sum_{t=1}^{T} \frac{E[FCF_{t}^{U}]}{(1 + w)^t} = V
\]

Thus the objective is to find the interest rate \( w \) which equates the total present value of all expected free cash flows to shareholders, assuming an unlevered business not using imputed equity tax shields (see Section 5.1), to the business value found earlier using either the equity method or the APV method. This implies that all effects induced by financing decisions (i.e., tax shields) are depicted exclusively in the interest rate.

In addition to the usual components of the WACC (e.g. capital structure, risk-free interest rate, risk-adjusted interest rates for equity holders), this WACC also includes growth effects and the tax benefit arising from imputed interest on increased equity.

**Proposition 10:** The WACC is calculated as follows:

\[
w = qROI + \frac{(1 + v)ROI(1 - \tau_k - q)\left(r_f - qROI\right)}{ROI\left[(1 + v)(1 - \tau_k) + qr_e\left(\tau_k - \tau_s\right) - q\right] - vr_f(1 - \tau_k) + v}
\]

This equation is derived in the Appendix.
WACC is the sum of the expected growth rate plus a non-negative adjustment which is a function of the risk-free interest rate, the market price of risk, the covariance $\rho$, the expected return on investment, the capital structure, the plowback ratio, the equity interest rate and the two tax rates.

In order to gain more insight into the influence of these factors, we have also performed a numerical sensitivity analysis. The following base case scenario is assumed: The book equity $EQ_0$ is 100, the risk-free interest rate $r_f$ is 3% p.a., the market price of risk (market risk premium per unit of variance in market return) is $\lambda = 5$, the covariance between the market return and the return on investment is $\rho = 0.0001$, and the expected return on investment is 12% p.a.. The capital structure $\nu$ is 1, the "plowback ratio" $q$ is 0.15, the fictitious imputed equity interest rate $r_e$ is 4% p.a., and the two tax rates are $\tau_K = 34 \%$ and $\tau_S = 25 \%$.

Using this base case scenario, each value was varied in a ceteris paribus analysis as follows:

$r_f$ between 2% p.a. and 7% p.a. at intervals of 1 percentage point
$\lambda$ between 1 and 10 at intervals of 1
$\rho$ between -0.0002 and 0.0002 at intervals of 0.00005
$\overline{ROI}$ between 5% and 17% at intervals of 2 percentage points
$\nu$ between 0 and 2.5 at intervals of 0.25
$q$ between 0.02 and 0.20 at intervals of 0.02
$r_e$ between 2% p.a. and 11% p.a. at intervals of 1 percentage point

Our analysis shows that $w$ is an increasing function of the risk-free rate $r_f$, the market price of risk $\lambda$, the covariance $\rho$ and the capital structure $\nu$. It is a decreasing function of the expected return on investment $\overline{ROI}$, the plowback ratio $q$ and the equity interest rate $r_e$. Holding (as in Chapter 6) the spread between the fictitious equity rate and the risk-free rate constant shows that $w$ is an increasing function of the risk-free rate.

All signs of the sensitivities except the one with respect to $\nu$ are inversely related to the signs of the sensitivities these parameters show in $V$ (see Chapter 6). Therefore the effect these parameters have on $V$ (which enters into the denominator of the weighted average cost of capital) is not overridden by the effect they have on the free cash flow $FCF_U^t$ (to be discounted by $w$). Only for $\nu$ does the effect the parameter has on $FCF_U^t$ change sign.

11 Non-negativity is a result of the restrictions from Chapter 6.
8 Summary

This paper is the first to integrate a tax system with imputed equity interest into business valuation. To this end, we present a value driver model for the example of a tax system, where the equity increase, not the equity holdings, is taken as the basis for imputed equity interest. Several methods of business valuation (equity method, APV method and entity method) are presented with attention to imputed equity interest. Due to the imputed equity interest, the APV method requires an adjustment for the tax benefit from equity tax shields in addition to the adjustment for the debt tax shields. Thus, an unlevered company which does not use equity tax shields has to be chosen as a common denominator in the application of the APV method. Furthermore, a closed-form solution for the value of equity tax shields is derived. On the basis of Fama (1977), the interest rate for the valuation of free cash flows and of the tax benefits arising from debt and equity financing are derived from the covariance between book return on investment and the market return. Inter-temporary differences in risk require the use of various risk-adjusted term structures on interest rates in the equity method as well as the APV method. A market-to-book ratio endogenous to the model thus arises. Finally, a proposal for determining the weighted average cost of capital in consideration of the deductibility of imputed equity interest is presented. A sensitivity analysis is conducted for all value components of a business and for the weighted average cost of capital.

The methodology we have applied to the Austrian example can also be used for other tax systems with imputed equity interest.
9 References


Wagner, F. and E. Wenger (1999), Was wir von Österreich lernen können, Handelsblatt from April 8th 1999.
Table 2: Growth in Book Equity of ATX Companies (Austrian Prime Market)\textsuperscript{12}

<table>
<thead>
<tr>
<th>Company</th>
<th>Growth p. a.</th>
</tr>
</thead>
<tbody>
<tr>
<td>AUA</td>
<td>17.46%</td>
</tr>
<tr>
<td>Austria Tabak</td>
<td>34.11%</td>
</tr>
<tr>
<td>Böhler</td>
<td>7.59%</td>
</tr>
<tr>
<td>BWT</td>
<td>14.40%</td>
</tr>
<tr>
<td>Erste</td>
<td>24.75%</td>
</tr>
<tr>
<td>Flughafen</td>
<td>9.65%</td>
</tr>
<tr>
<td>Generali</td>
<td>4.85%</td>
</tr>
<tr>
<td>Mavry Melnhof</td>
<td>8.77%</td>
</tr>
<tr>
<td>OMV</td>
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</tr>
<tr>
<td>RHI</td>
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</tr>
<tr>
<td>VA-Tech</td>
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<tr>
<td>Verbund</td>
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<td>Wienerberger</td>
<td>11.79%</td>
</tr>
<tr>
<td>Wolford</td>
<td>15.48%</td>
</tr>
</tbody>
</table>

Proof - Proposition 1 (Interest Rates - Equity Method):

Interest rates for the equity method, which are denoted by $k_{E(s,t)}$, are derived using the methodology presented by Fama (1977):

The free cash flow at $t - FCF_t$ - is known to be:

$$E_{Q_s-1}[ROI_t [(1 + \nu)(1 - \tau_K) + q_r(\tau_K - \tau_s) - q] - v_r (1 - \tau_K)]$$

We begin with an analysis for cases where $s < t$:

First the variable $\varepsilon_s$ is defined as follows:

$$\varepsilon_s = \frac{E_s[EQ_{s-1}][ROI_t [(1 + \nu)(1 - \tau_K) + q_r(\tau_K - \tau_s) - q] - v_r (1 - \tau_K)]}{E_s[EQ_{s-1}][ROI_t [(1 + \nu)(1 - \tau_K) + q_r(\tau_K - \tau_s) - q] - v_r (1 - \tau_K)]} - 1$$

Because $EQ_{s-1}$ and $ROI_t$ are uncorrelated, this can be rewritten/expressed as follows:

$$\varepsilon_s = \frac{E_s[EQ_{s-1}][E_s[ROI_t [(1 + \nu)(1 - \tau_K) + q_r(\tau_K - \tau_s) - q] - v_r (1 - \tau_K)]]}{E_s[EQ_{s-1}][ROI_t [(1 + \nu)(1 - \tau_K) + q_r(\tau_K - \tau_s) - q] - v_r (1 - \tau_K)]} - 1$$

$$\varepsilon_s = \frac{E_s[EQ_{s-1}][ROI_t [(1 + \nu)(1 - \tau_K) + q_r(\tau_K - \tau_s) - q] - v_r (1 - \tau_K)]}{E_s[EQ_{s-1}][ROI_t [(1 + \nu)(1 - \tau_K) + q_r(\tau_K - \tau_s) - q] - v_r (1 - \tau_K)]} - 1$$

$$\varepsilon_s = \frac{E_s[EQ_{s-1}][ROI_t [(1 + \nu)(1 - \tau_K) + q_r(\tau_K - \tau_s) - q] - v_r (1 - \tau_K)]}{E_s[EQ_{s-1}][ROI_t [(1 + \nu)(1 - \tau_K) + q_r(\tau_K - \tau_s) - q] - v_r (1 - \tau_K)]} - 1$$

$$\varepsilon_s = \frac{E_s[EQ_{s-1}][ROI_t [(1 + \nu)(1 - \tau_K) + q_r(\tau_K - \tau_s) - q] - v_r (1 - \tau_K)]}{E_s[EQ_{s-1}][ROI_t [(1 + \nu)(1 - \tau_K) + q_r(\tau_K - \tau_s) - q] - v_r (1 - \tau_K)]} - 1$$

$$\varepsilon_s = \frac{E_s[EQ_{s-1}][ROI_t [(1 + \nu)(1 - \tau_K) + q_r(\tau_K - \tau_s) - q] - v_r (1 - \tau_K)]}{E_s[EQ_{s-1}][ROI_t [(1 + \nu)(1 - \tau_K) + q_r(\tau_K - \tau_s) - q] - v_r (1 - \tau_K)]} - 1$$

The covariance between $\varepsilon_s$ and the market return in the same year, $R_{M_t}$, is:

$$Cov[\varepsilon_s, R_{M_t}] = \frac{\rho q}{1 + q R_{O_l}}$$

This results in the following interest rate:

\textsuperscript{12} Source: Reuters Information Systems
\[ k_E(s,t) = \frac{1 + r_f}{1 - \lambda \cdot \text{Cov}[e_s, r_{m_s}]} - 1 = \frac{1 + r_f}{1 - \frac{\lambda q \rho}{1 + q \text{ROI}}} - 1 = \frac{1 + r_f}{1 + q \text{ROI} - \lambda q \rho} - 1 = (1 + r_f)^{1 + q \text{ROI}} - 1 \]

where \( \text{ROI} \) denotes \( \frac{\text{ROI} - \lambda \rho}{\cdot} \).

A similar analysis for the case where \( s = t \) follows:

The variable \( e_{ts} \) is defined in an manner analogous to the case where \( s < t \):

\[ e_{ts} = \frac{E_{s-1}[\text{ROI}_s [(1 + v)(1 - \tau_k) + q r_c(\tau_k - \tau_s) - q] - r f(1 - \tau_k)]}{E_{s-1}[\text{ROI}_s [(1 + v)(1 - \tau_k) + q r_c(\tau_k - \tau_s) - q] - r f(1 - \tau_k)]} - 1 = \frac{\text{ROI}_s [(1 + v)(1 - \tau_k) + q r_c(\tau_k - \tau_s) - q] - r f(1 - \tau_k)]}{\text{ROI}_s [(1 + v)(1 - \tau_k) + q r_c(\tau_k - \tau_s) - q] - r f(1 - \tau_k)]} - 1 = \frac{\text{ROI}_s [(1 + v)(1 - \tau_k) + q r_c(\tau_k - \tau_s) - q] - r f(1 - \tau_k)}{\text{ROI}_s [(1 + v)(1 - \tau_k) + q r_c(\tau_k - \tau_s) - q] - r f(1 - \tau_k)} - 1 \]

The covariance between \( e_{ts} \) and the market return in the same year is:

\[ \text{Cov}[e_{ts}, r_{m_s}] = \frac{\rho [(1 + v)(1 - \tau_k) + q r_c(\tau_k - \tau_s) - q]}{\text{ROI}_s [(1 + v)(1 - \tau_k) + q r_c(\tau_k - \tau_s) - q] - r f(1 - \tau_k)} \]

The following interest rate results:

\[ k_E(s,t) = \frac{1 + r_f}{1 - \lambda \cdot \text{Cov}[e_{ts}, r_{m_s}]} - 1 = \frac{1 + r_f}{1 - \frac{\lambda q \rho}{\text{ROI}_s [(1 + v)(1 - \tau_k) + q r_c(\tau_k - \tau_s) - q] - r f(1 - \tau_k)]} - 1 = \frac{1 + r_f}{\text{ROI}_s [(1 + v)(1 - \tau_k) + q r_c(\tau_k - \tau_s) - q] - r f(1 - \tau_k)} - 1 = \frac{1 + r_f}{\text{ROI}_s [(1 + v)(1 - \tau_k) + q r_c(\tau_k - \tau_s) - q] - r f(1 - \tau_k)} - 1 = \frac{1 + r_f}{\text{ROI}_s [(1 + v)(1 - \tau_k) + q r_c(\tau_k - \tau_s) - q] - r f(1 - \tau_k)} - 1 = \frac{1 + r_f}{\text{ROI}_s [(1 + v)(1 - \tau_k) + q r_c(\tau_k - \tau_s) - q] - r f(1 - \tau_k)} - 1 = \frac{1 + r_f}{\text{ROI}_s [(1 + v)(1 - \tau_k) + q r_c(\tau_k - \tau_s) - q] - r f(1 - \tau_k)} - 1 = \frac{1 + r_f}{\text{ROI}_s [(1 + v)(1 - \tau_k) + q r_c(\tau_k - \tau_s) - q] - r f(1 - \tau_k)} - 1 \]

**Proof - Proposition 2 (Equity Value - Equity Method):**

The value of free cash flow in year \( t \) is:

\[ PV(FCF_t) = E K_0 [(1 + q \text{ROI})^{-1} \text{ROI}_t [(1 + v)(1 - \tau_k) + q r_c(\tau_k - \tau_s) - q] - r f(1 - \tau_k)]] \]

The value of all free cash flows to equity holders is to be found. Assuming an infinite life span, this is done using the summation formula for the infinite geometric series:

The first term (present value of the first payment) is:

\[ PV(FCF_t) = E K_0 [(1 + q \text{ROI})^{-1} \text{ROI}_t [(1 + v)(1 - \tau_k) + q r_c(\tau_k - \tau_s) - q] - r f(1 - \tau_k)]] \]
The growth factor for the present value of the payment is:

\[
z = \frac{1 + q\text{ROI}}{1 + r_f}
\]

and

\[
1 - z = 1 - \frac{1 + q\text{ROI}}{1 + r_f} = \frac{r_f - q\text{ROI}}{1 + r_f}
\]

Thus:

\[
V_{EK} = \frac{\text{PV}(FCF_t)}{(1 - z)} = \frac{EQ_0 \left[ \text{ROI} \left \{ (1 + \nu)(1 - \tau_K) + qr_s (\tau_s - \tau_s) - \nu r_f (1 - \tau_K) \right \} \right ]}{r_f - q\text{ROI}}
\]

**Proof - Proposition 3** (Interest Rates for the Unlevered Company not using Equity Tax Shields):

We begin with an analysis for cases where \( s < t \):

\[
\varepsilon_{st} = \frac{EQ_s \left \{ FCF_t^{U} \right \}}{E_{s-1} \left \{ FCF_t^{U} \right \}} - 1 = \frac{EQ_s \left \{ FCF_t^{U} \right \}}{E_{s-1} \left \{ FCF_t^{U} \right \}} - 1 = \frac{EQ_s \left \{ (1 + q\text{ROI})^{-s-1} \right \} - 1}{EQ_{s-1} \left \{ (1 + q\text{ROI})^{-s} \right \}}
\]

Because \( \varepsilon_{st} \) is identical to its counterpart under the equity method, the covariance between \( \varepsilon_{st} \) and the market return - and thus the interest rate - are likewise identical.

A similar analysis for the case where \( s = t \) follows:

\[
\varepsilon_{st} = \frac{EQ_s \left \{ (1 - \tau_K - q) \right \}}{E_{s-1} \left \{ (1 - \tau_K - q) \right \}} - 1 = \frac{ROIs(1 - \tau_K - q)}{ROI(1 - \tau_K - q)} - 1 = \frac{ROIs}{ROI} - 1
\]

The covariance between \( \varepsilon_{st} \) and the market return in the same year is:

\[
\text{COV}[\varepsilon_{st}, r_{M,t}] = \frac{\rho}{\text{ROI}}
\]

The interest rate can be calculated as in the proof for Proposition 1:

\[
k_U(s,t) = \frac{1 + r_f}{1 - \frac{\rho}{\text{ROI}}} - 1 = \frac{1 + r_f}{\text{ROI}} - 1 = \frac{ROIs}{ROI} - 1
\]

**Proof - Proposition 4** (Value of Unlevered Company not using Equity Tax Shields - APV Method):

The value of free cash flow in year \( t \) for the unlevered company is:

\[
PV(FCF_t^{U}) = \frac{EQ_t^{U} \left \{ (1 + q\text{ROI})^{-t} \right \} \text{ROI}(1 - \tau_K - q)}{(1 + r_f)}
\]

Under the assumption of an infinite life span, the value of the unlevered company is found using the summation formula for the geometric series:

\[
V^U = \frac{PV(FCF_t^{U})}{(1 - z)}
\]
where \( PV(FCF^U) = \frac{EO^U ROI (1 - \tau_K - q)}{1 + r_f} \)
and \( z \) is defined as in the equity method. The following thus results:
\[
V^U = \frac{EO^U ROI (1 - \tau_K - q)}{r_f - q ROI}
\]

**Proof - Proposition 5 (Interest Rate for Debt Tax Shields):**

We begin with an analysis for cases where \( s < t \):

\[
\varepsilon_s = \frac{E_s[TSD_s]}{E_s[TSD_s]} - 1 = \frac{E_s[EO_s, \nu, \tau_K]}{E_s[EO_s, \nu, \tau_K]} - 1 = \frac{EO_s}{EO_s (1 + q ROI)} - 1 = \frac{1 + q ROI}{1 + q ROI} - 1
\]

Because \( \varepsilon_s \) is identical to its counterpart under the equity method, the covariance and thus the interest rate are likewise identical.

A similar analysis for the case where \( s = t \) follows:

\[
\varepsilon_s = \frac{EO_{s-1}, \nu, \tau_K}{EO_{s-1}, \nu, \tau_K} - 1 = 0
\]

Thus the covariance with the market return is 0, meaning that the risk-free interest rate is to be applied.

**Proof - Proposition 6 (Value of Debt Tax Shields - APV Method):**

The value of the debt tax shields in year \( t \) is:

\[
PV(TSD_t) = \frac{EO_0 (1 + q ROI)^{-1} \nu, \tau_K}{(1 + r_f)}
\]

Under the assumption of an infinite life span, the value of the debt tax shields using the summation formula for the geometric series is:

\[
PV(TSD) = \frac{PV(TSD_t)}{1 - z}
\]

where \( PV(TSD_t) = \frac{EO_0 \nu, \tau_K}{1 + r_f} \) and \( z \) is defined as above. The following then results:

\[
PV(TSD) = \frac{EO_0 \nu, \tau_K}{r_f - q ROI}
\]

**Proof - Proposition 7 (Interest Rate for Equity Tax Shields):**

We begin with an analysis for cases where \( s < t \):

\[
\varepsilon_s = \frac{E_s[TSE_s]}{E_s[TSE_s]} - 1 = \frac{E_s[EO_{s-1}, \nu, \tau_K, q, (\tau_K - \tau_s)]}{E_s[EO_{s-1}, \nu, \tau_K, q, (\tau_K - \tau_s)]} - 1 = \frac{E_s[EO_{s-1}]}{E_s[EO_{s-1}]} - 1 = \frac{1 + q ROI}{1 + q ROI} - 1
\]
A similar analysis for the case where \( s = t \) follows:

\[
\varepsilon_{ts} = \frac{E_{s-1} \cdot ROI_q \cdot (\tau_K - \tau_s)}{E_{s-1} \cdot [E_{s-1} \cdot ROI_q \cdot (\tau_K - \tau_s)]} - 1 = \frac{ROI_q}{ROI} - 1
\]

Because in both cases \( \varepsilon_{ts} \) is identical to its counterpart in the unlevered company, the covariance and thus the interest rates are likewise identical.

**Proof - Proposition 8** (Value of Equity Tax Shields - APV Method):

The value of the tax shield arising from imputed equity interest in year \( t \) is:

\[
PV(TSE_t) = \frac{E_{t} \cdot ROI_q \cdot (\tau_K - \tau_s)}{(1 + q ROI)(1 + r_f)}
\]

Under the assumption of an infinite life span, the value of the equity tax shields is calculated using the summation formula for the geometric series as follows:

\[
PV(TSE) = \frac{PV(TSE_t)}{(1 - z)}
\]

where \( PV(TSE_t) = \frac{E_{t} \cdot ROI_q \cdot (\tau_K - \tau_s)}{1 + r_f} \)

The following results:

\[
PV(TSE) = \frac{E_{0} \cdot ROI_q \cdot (\tau_K - \tau_s)}{r_f - q ROI}
\]

**Proof - Proposition 9** (Equivalence of Value from Equity Method and APV Method):

Here the objective is to demonstrate that the sum of the three components from the APV method corresponds to the value found under the equity method. The value of the business under the equity method is equal to the value of equity plus the value of debt.

\[
V = V_{EQ} + D_0
\]

\[
V = \frac{E_{0} \cdot ROI \cdot (1 + \nu)(1 - \tau_K) + q \cdot (\tau_K - \tau_s) - q \cdot \nu \cdot (1 - \tau_K) + E_{0} \cdot v \cdot r_f - q ROI}{r_f - q ROI}
\]

The three components from the APV method are as follows:

The value of the unlevered business is:

\[
V^U = \frac{E_{0} \cdot ROI \cdot (1 - \tau_K - q)}{r_f - q ROI}, \text{ or, because } EK^U_0 = E_{0} \cdot (1 + \nu)
\]

\[
V^U = \frac{E_{0} \cdot (1 + \nu)ROI \cdot (1 - \tau_K - q)}{r_f - q ROI}
\]

The value of the debt tax shields is:

\[
PV(TSD) = \frac{E_{0} \cdot v \cdot r_f \cdot \tau_K}{r_f - q ROI}
\]

The value of the equity tax shields is:
\[ PV(TSE) = \frac{E_Q \overline{ROI} q r_y (\tau_K - \tau_S)}{r_y - q ROI} \]

When these three components are added and then equated to the value of the business under the equity method, \( \frac{E_Q \overline{ROI}}{r_y - q ROI} \) cancels out. Replacing all remaining \( ROI \) by \( \overline{ROI} - \lambda \rho \) gives:

\[
(\overline{ROI} - \lambda \rho) \left[ (1 + v)(1 - \tau_K) + q r_s (\tau_K - \tau_S) - q \right] - v r_y (1 - \tau_K) + v [r_f - q (\overline{ROI} - \lambda \rho)] =
\]

\[= (1 + v) (\overline{ROI} - \lambda \rho) (1 - \tau_K - q) + v r_y (\tau_K - q) + q r_s (\tau_K - \tau_S) \overline{ROI} - \lambda \rho \]

Because all components which contain the risk-free interest rate cancel each other out, the result is:

\[
(\overline{ROI} - \lambda \rho) \left[ (1 + v)(1 - \tau_K) + q r_s (\tau_K - \tau_S) - q \right] - v q (\overline{ROI} - \lambda \rho) =
\]

\[= (1 + v) (\overline{ROI} - \lambda \rho) (1 - \tau_K - q) + q r_s (\tau_K - \tau_S) \overline{ROI} - \lambda \rho \]

If the equation is then divided by \( (\overline{ROI} - \lambda \rho) \), the result is:

\[ (1 + v)(1 - \tau_K) + q r_s (\tau_K - \tau_S) - q = (1 + v)(1 - \tau_K - q) + q r_s (\tau_K - \tau_S) \]

Then \( q r_s (\tau_K - \tau_S) \) is subtracted from both sides.

\[ (1 + v)(1 - \tau_K - q) = (1 + v)(1 - \tau_K - q) \]

Q.E.D.

Thus the APV method and the equity method deliver the same results.

**Proof - Proposition 10 (WACC Calculation under the Entity Method)**

The WACC is defined as follows:

\[ \lim_{t \to \infty} \sum_{t=1}^{T} \frac{E[FCF_{it}^U]}{(1 + w)^t} = V \]

The expected free cash flow in year \( t \) for the unlevered company is, as we know from Section 5.1, \( E[FCF_{it}^U] = EQ_0 (1 + q \overline{ROI})^{T-i} (1 + v) \overline{ROI} (1 - \tau_K - q) \). Thus the term for the year \( t \) is:

\[ EQ_0 (1 + q \overline{ROI})^{T-i} (1 + v) \overline{ROI} (1 - \tau_K - q) \]

\[\frac{(1 + v) \overline{ROI} (1 - \tau_K - q)}{(1 + w)(1 + w)^{-1}} \]

The term for the first year is:

\[ EQ_0 (1 + v) \overline{ROI} (1 - \tau_K - q) \]

\[\frac{1 + w}{1 + w} \]

The total present value under the assumption of an infinite life span can again be calculated using the exponential series. The growth factor for this series is:

\[ y = \frac{1 + q \overline{ROI}}{1 + w} \]

so that \( 1 - y = \frac{w - q \overline{ROI}}{1 + w} \)

Thus the total present value of all free cash flows is:

\[ EQ_0 (1 + v) \overline{ROI} (1 - \tau_K - q) \]

\[\frac{w - q \overline{ROI}}{w - q \overline{ROI}} \]

This value is then equated to the value of the business under the equity method, \( V \):

\[ V = EQ_0 (1 + v) \overline{ROI} (1 - \tau_K - q) \]

\[\frac{w - q \overline{ROI}}{w - q \overline{ROI}} \]
Thus

\[ w = q \overline{ROI} + \frac{EQ_0 (1+v) \overline{ROI} (1 - \tau_K - q)}{V} = \]

\[ = q \overline{ROI} + \frac{(1+v) \overline{ROI} (1 - \tau_K - q) \left( q_f - q \overline{ROI} \right)}{\overline{ROI} \left[ (1+v)(1-\tau_K) + qr_f (\tau_K - \tau_s) - q_f \right] - wr_f (1-\tau_K) + v} \]